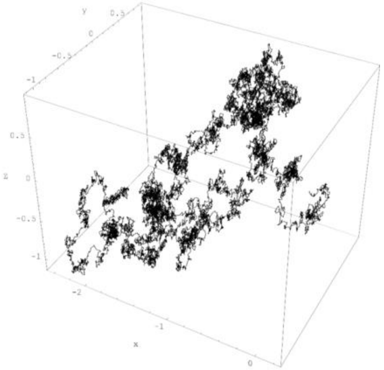
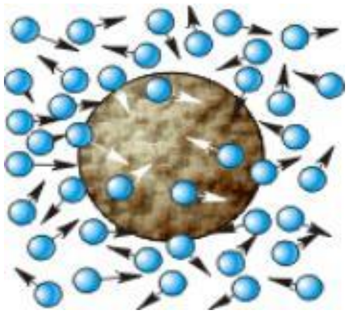
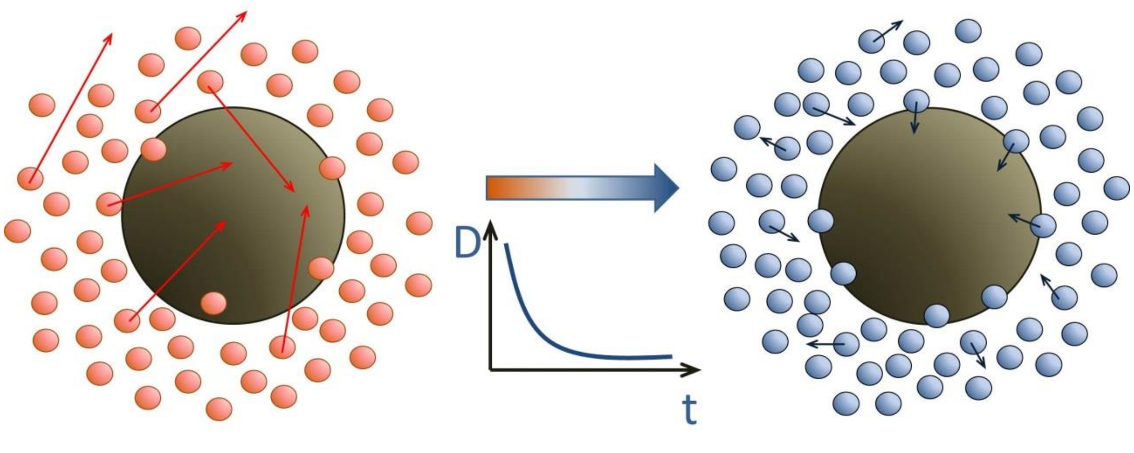




Brownian motion with time-dependent diffusion coefficient

Anna Bodrova

Skolkovo Institute of Science and Technology



Acknowledgement

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University of Leicester,
United Kingdom

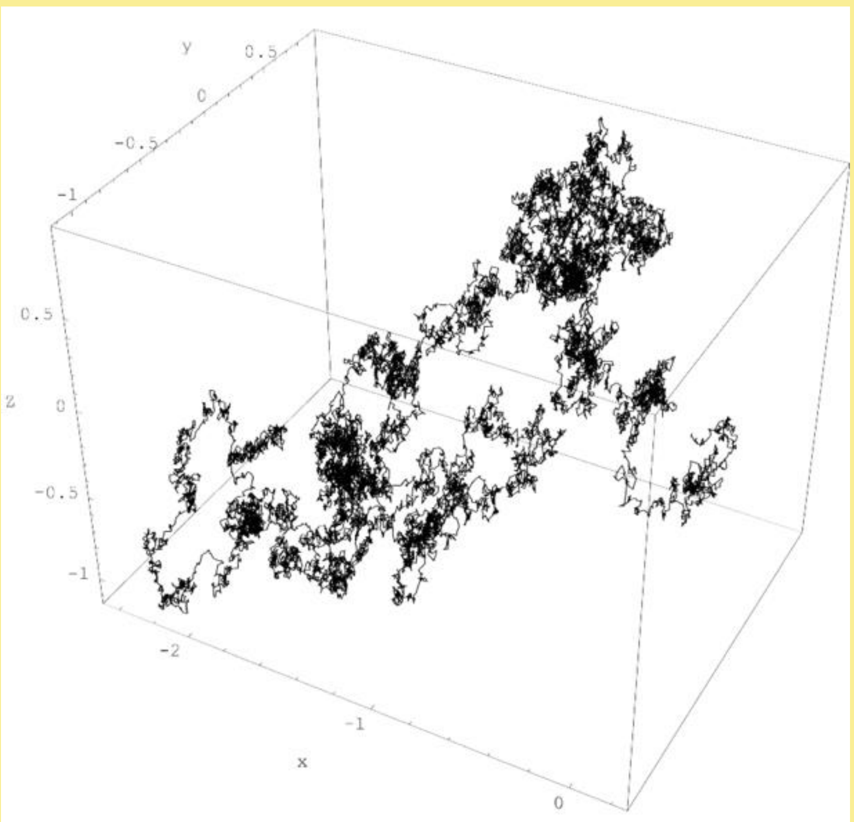
Igor Sokolov
Humboldt University,
Berlin, Germany

Normal diffusion

A. Einstein (1905)

$$\left\langle R^2(\mathrm{t}) \right\rangle = 6Dt$$

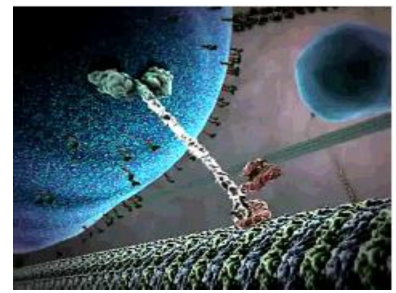
D – diffusion coefficient

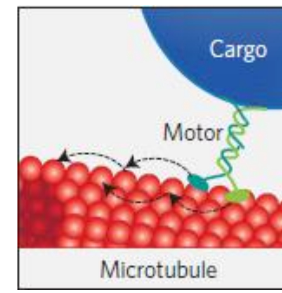


Anomalous diffusion

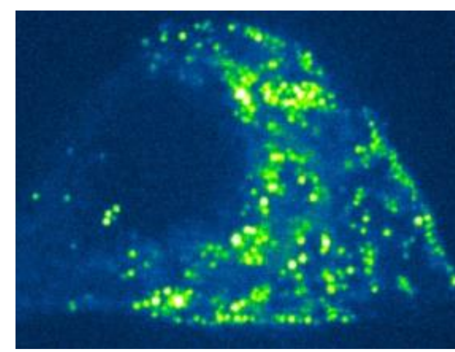
$$\langle R^2(\mathrm{t})\rangle \sim t^{\alpha} \qquad \alpha \neq 1$$

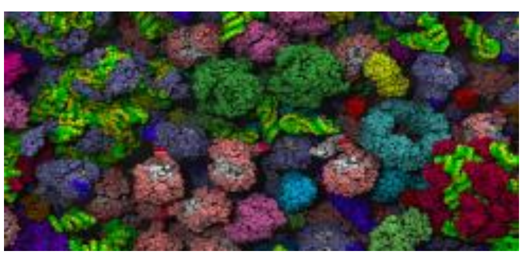
$\alpha > 1$ *Superdiffusion*

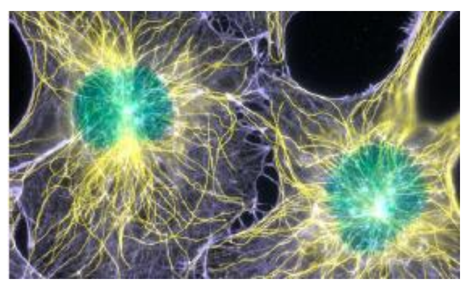




$\alpha < 1$ *Subdiffusion*







$\alpha = 1$ *Normal diffusion*

Anomalous diffusion in biology

Superdiffusion

- Flight of albatroses.
- Movement of spider monkeys.
- Molecular motors: active motion of motor proteins with cargo along the filaments in the cytoskeleton.

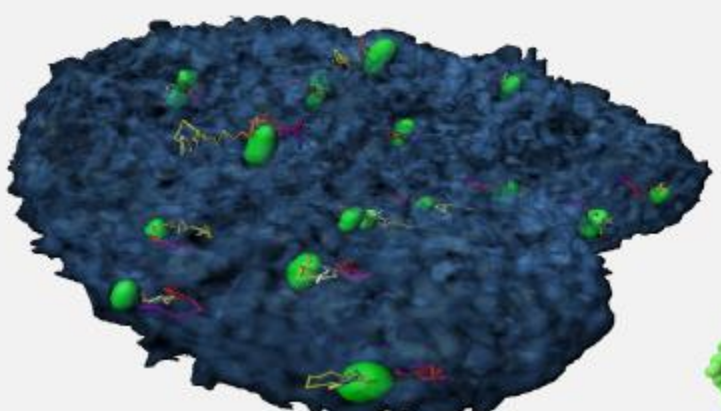
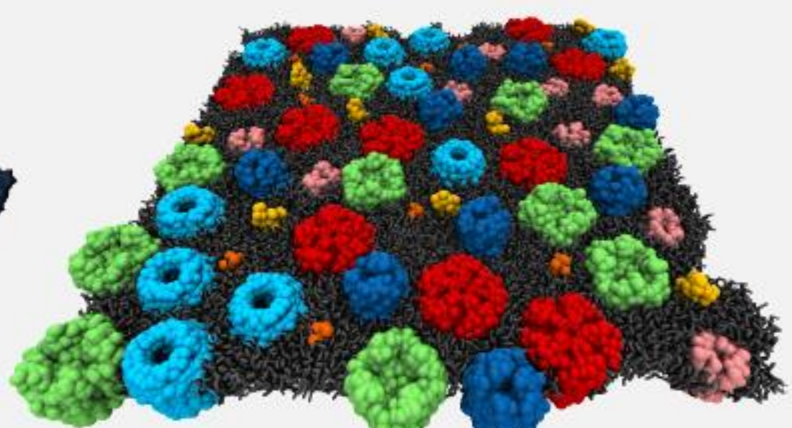
Advantages of superdiffusion in biology

- Superdiffusion leads to the effective search strategy for finding randomly located objects.

Anomalous diffusion in biology

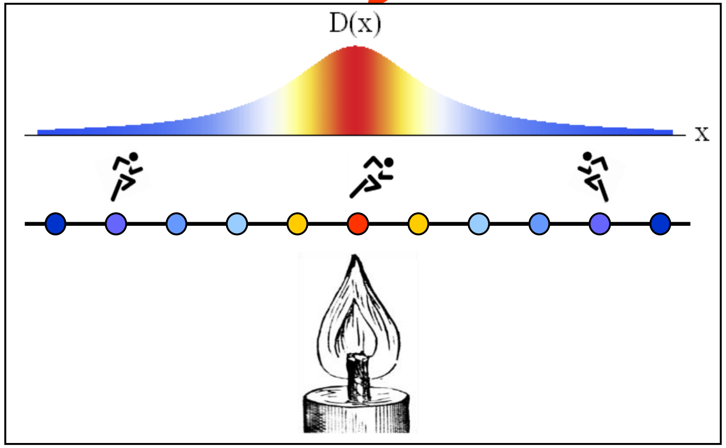
Subdiffusion

- Diffusion of channel proteins in cell membranes.
- Diffusion of telomers (chromosomal end parts) in human cell nuclei.
- Motion of messenger RNA in bacteria cells.
- Anomalous diffusion of large molecules due to high density of the cell environment.

Some sources of anomaly

Geometrical constraints



- Diffusion in crowded systems
- Diffusion on fractal structures

Diffusion in inhomogeneous environment

- Heterogeneous diffusion process (motion with space-dependent diffusion coefficient)

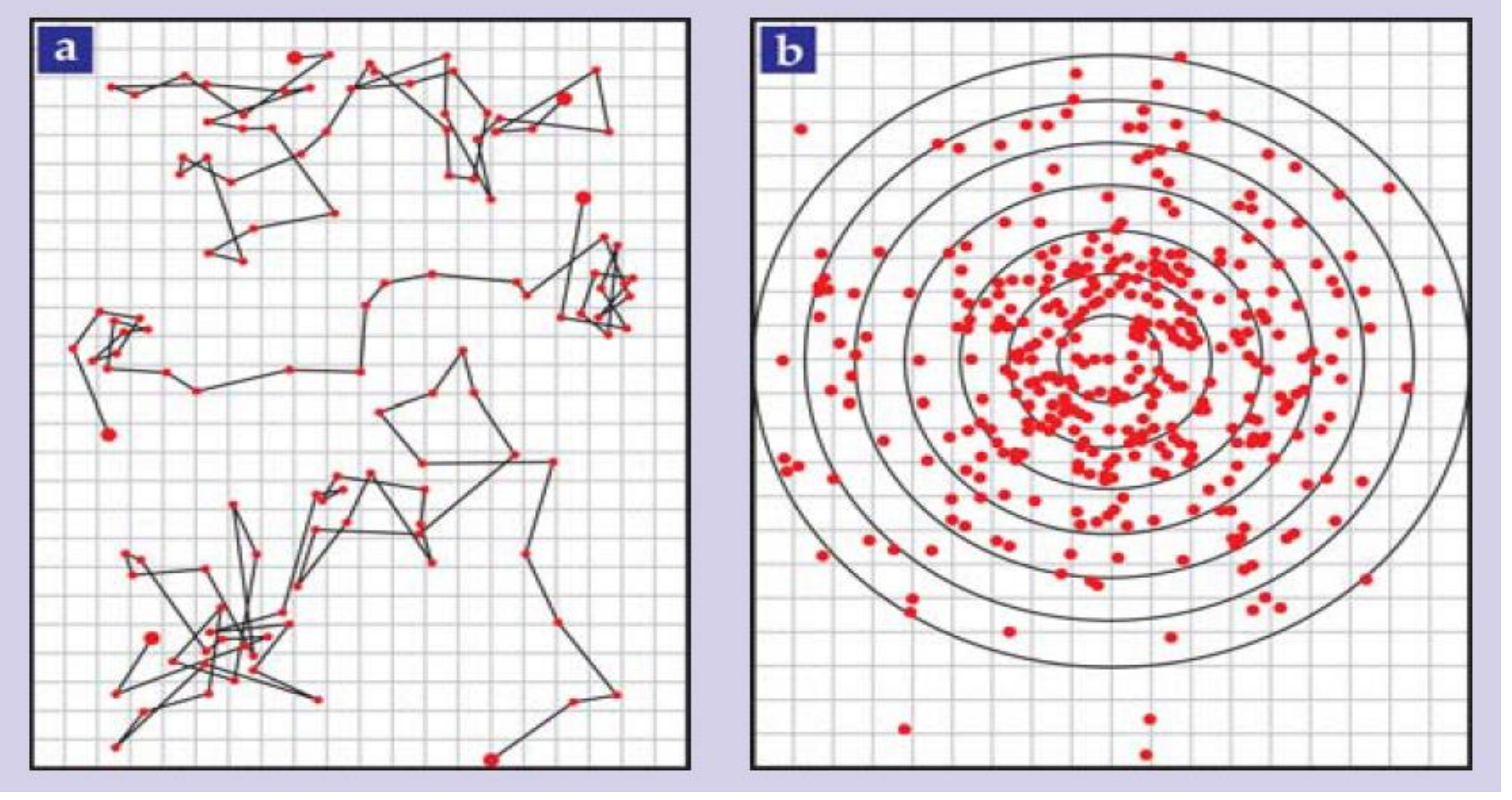
Diffusion in non-stationary environment

- Scaled Brownian motion (motion with time-dependent diffusion coefficient)

Ergodicity breaking

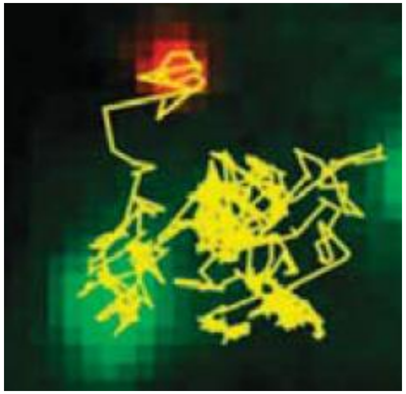
Ensemble averaged mean-squared displacement (MSD) is not equal to the ensemble-averaged MSD.

$$\left\langle \overline{\delta^2(\Delta)} \right\rangle \neq \left\langle x^2(\Delta) \right\rangle$$



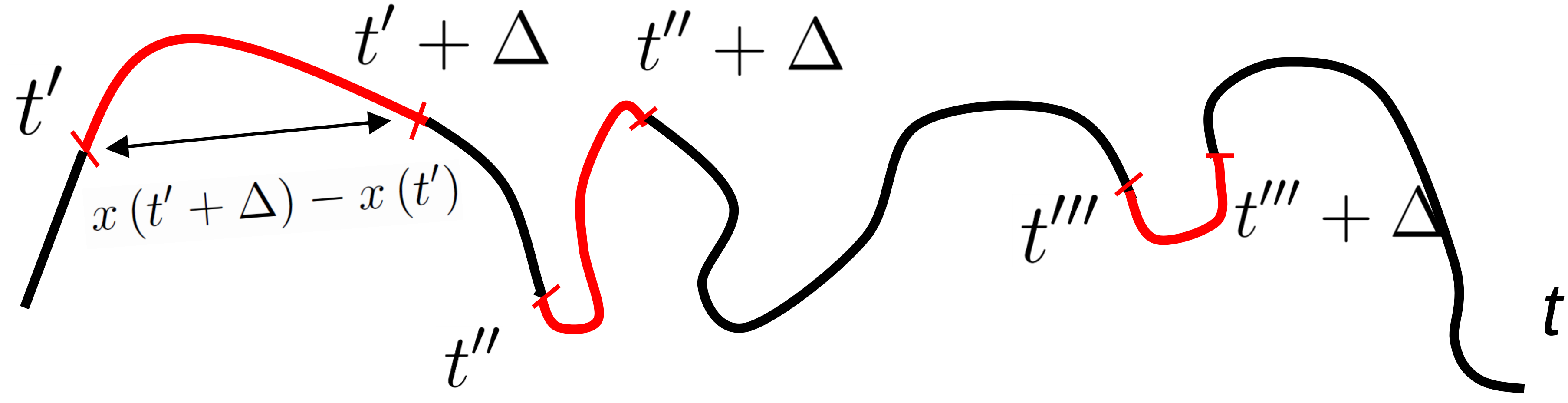
Time-averaged mean-squared displacement (MSD)

Single particle tracking experiments

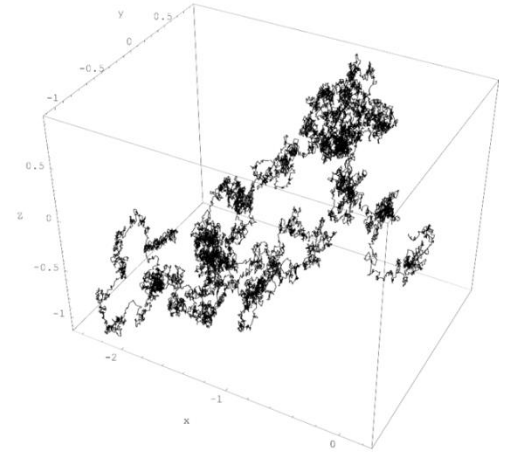


$$\overline{\delta^2(\Delta)} = \frac{1}{t-\Delta}\int_0^{t-\Delta} \left[x(t'+\Delta) - x(t')\right]^2 dt'$$

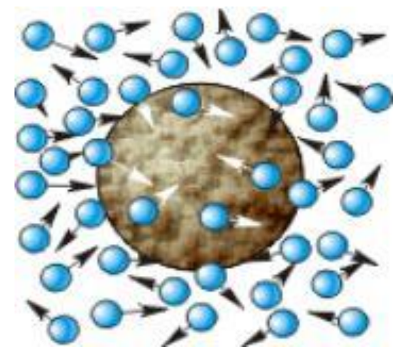
Δ – lag time *t – trajectory length*



Time-dependent diffusion coefficient D(t)



I. Brownian motion in a bath with time-dependent temperature



II. Snow melt dynamics

III. Diffusion in turbulence

IV. Water diffusion in brain tissue

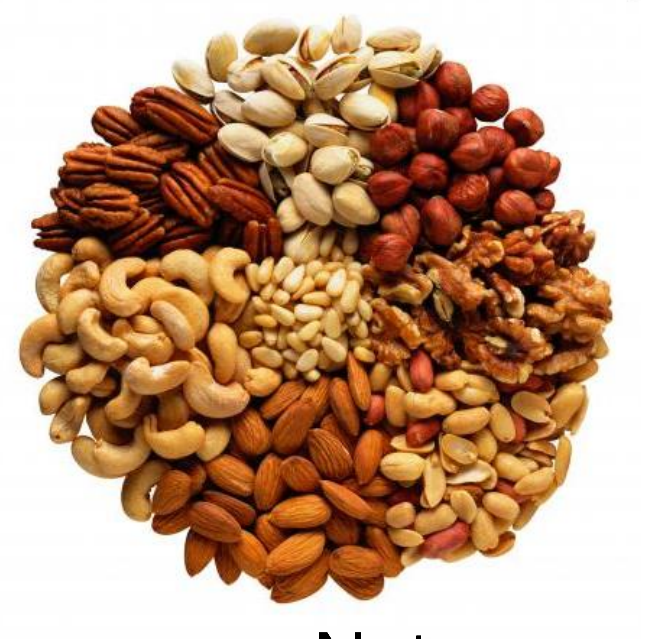
V. Free cooling granular gases

Granular systems

Powders

Sand

Stones



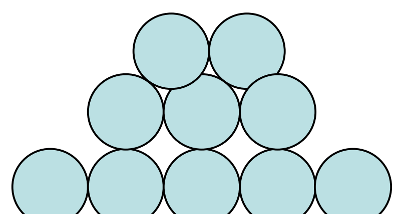
Nuts

Granular solids

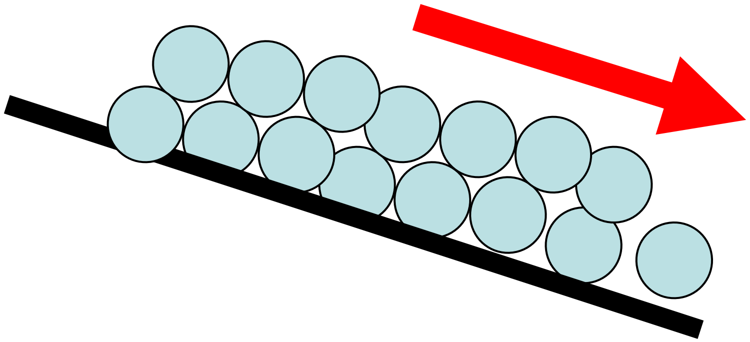
For small loads granular systems behave like solids: resist the load and preserve their form

Sand

Stones

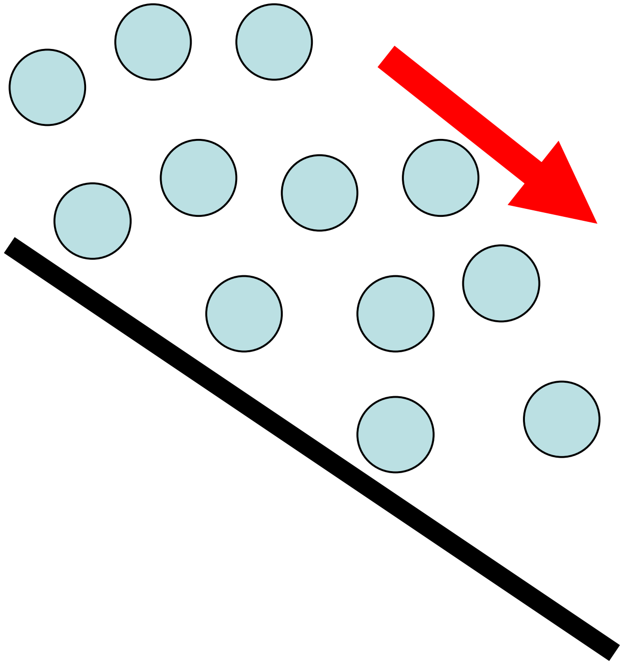


Granular liquids

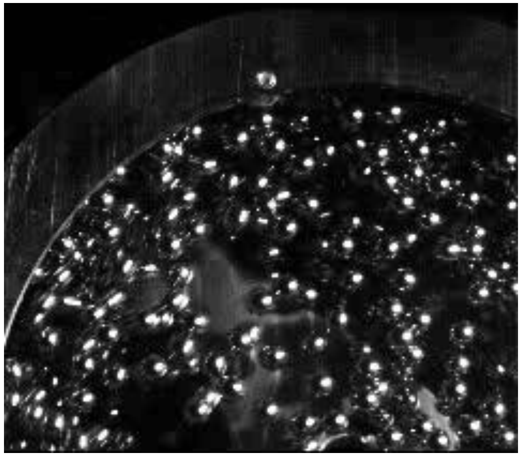
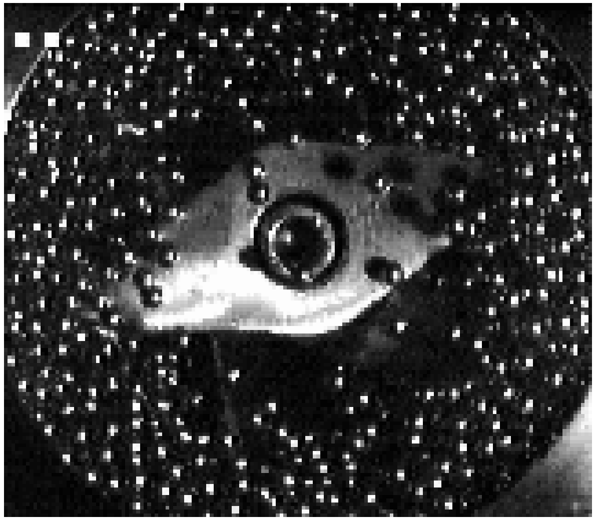


For lager loads they flow like liquids and preserve their volume

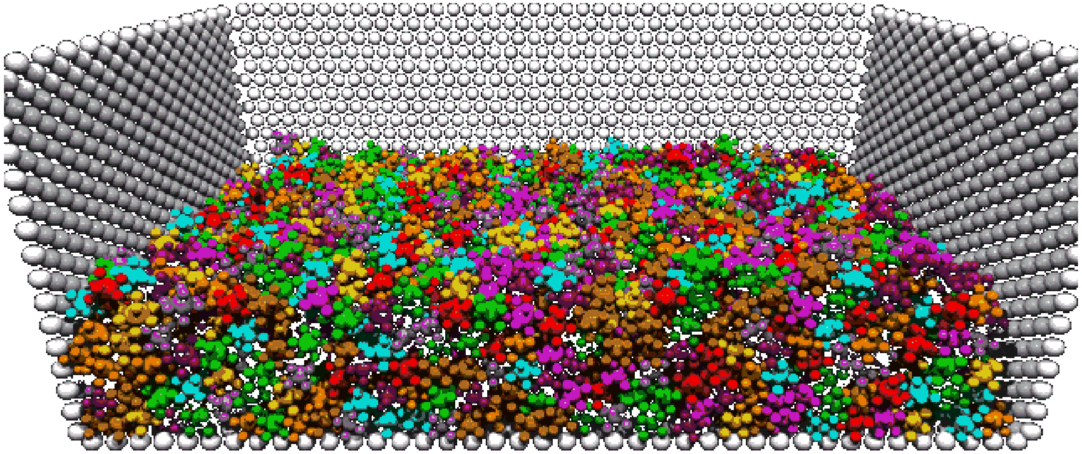
Granular gases



For larger loads they
form rapid granular flows,
where they behave like gases

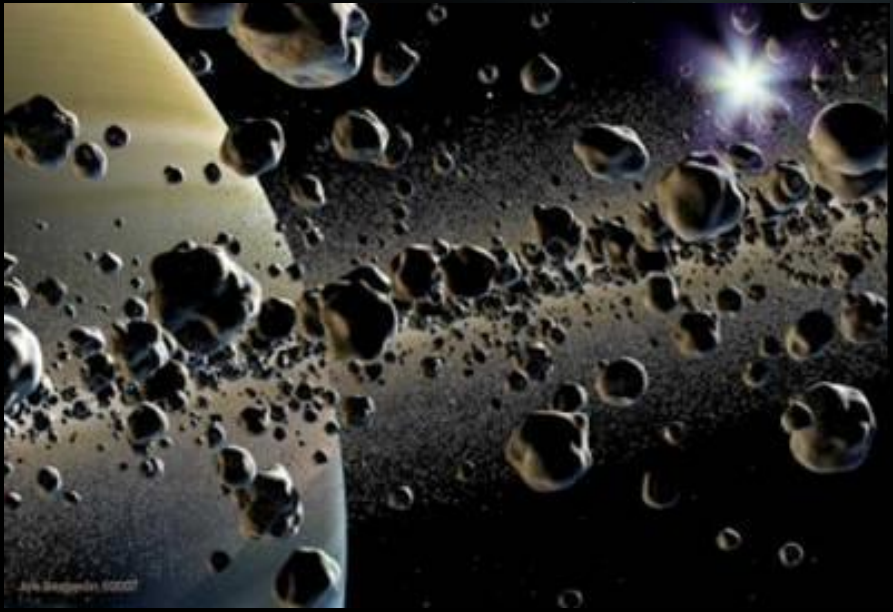




Granular gas – *rarefied* *system of* *macroscopic* *particles, which collide with* *loss of energy*

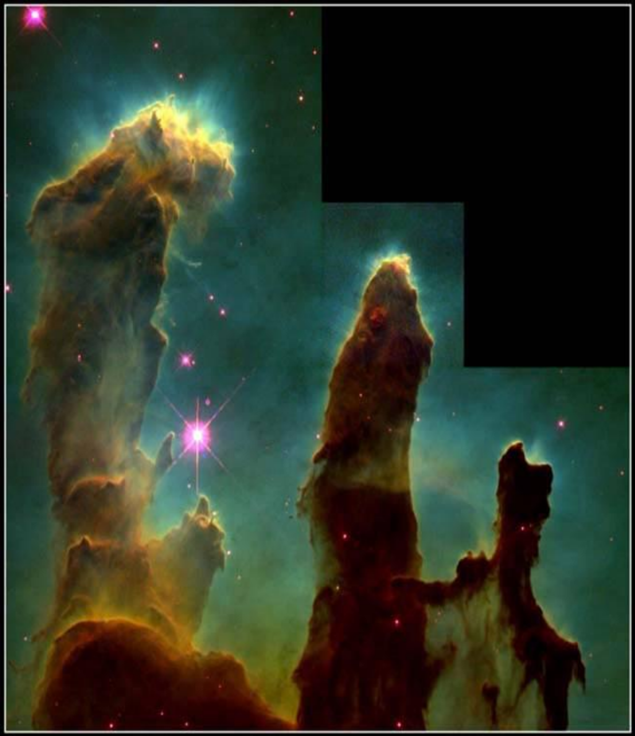



Granular gas: space

Protoplanetary disc

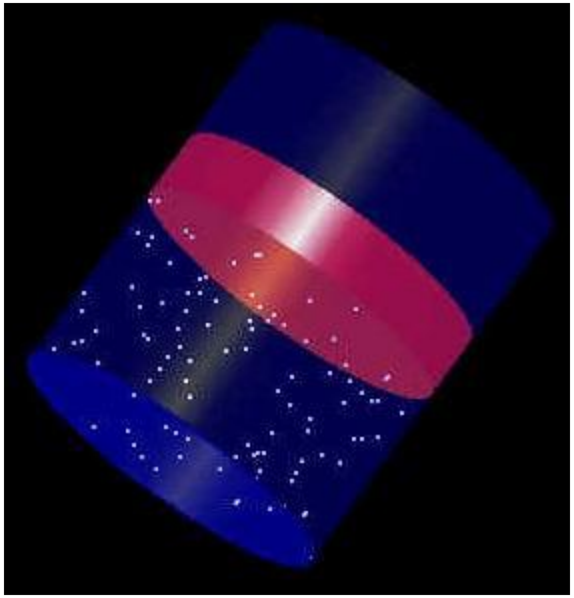


Planetary rings

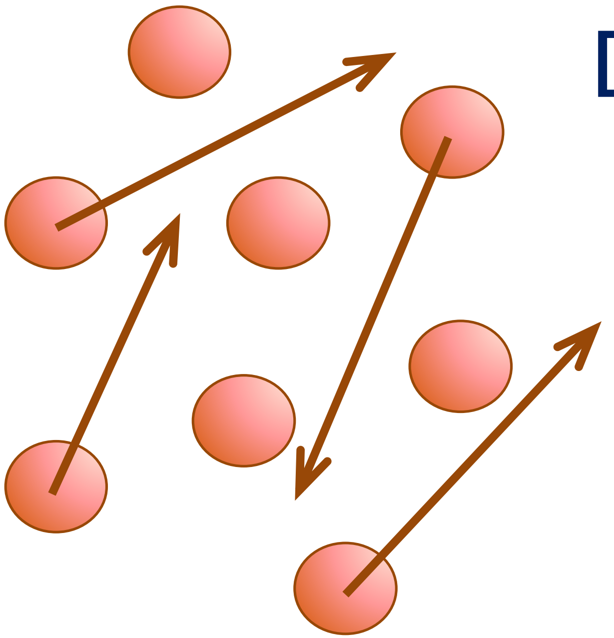


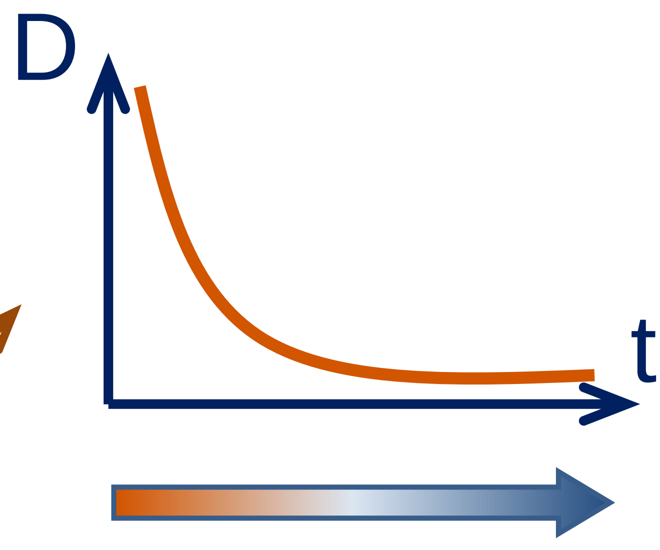
Interstellar dust

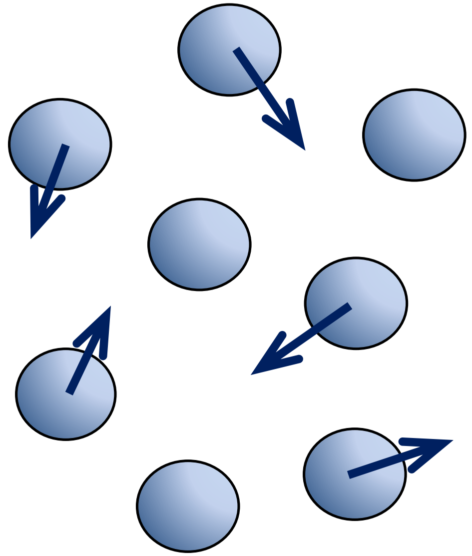
Free granular gas



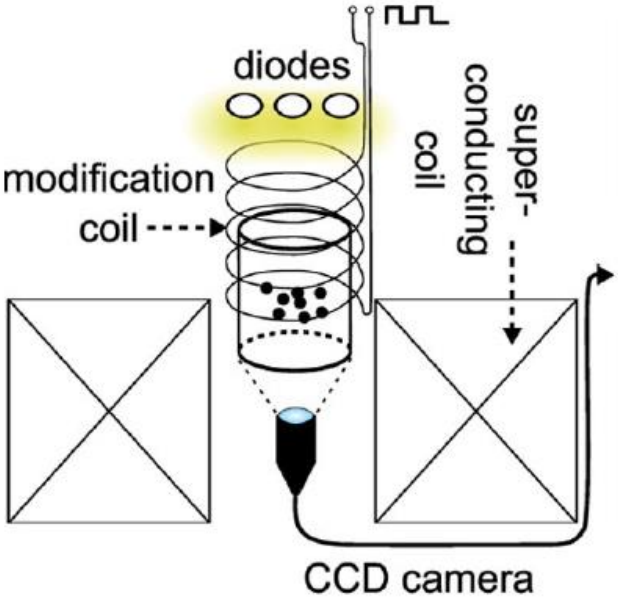
If no external forces act on the granular system, it evolves freely and the particles slow down.

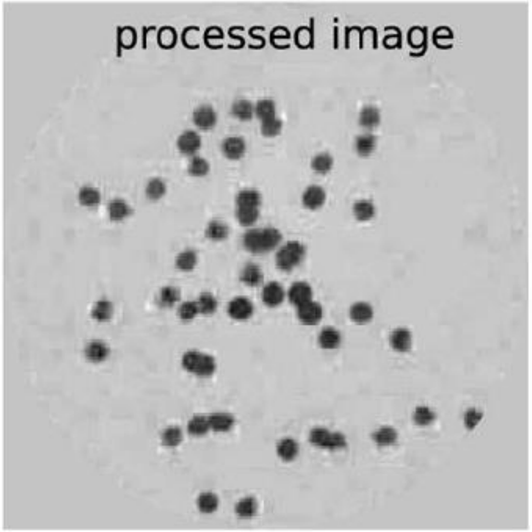






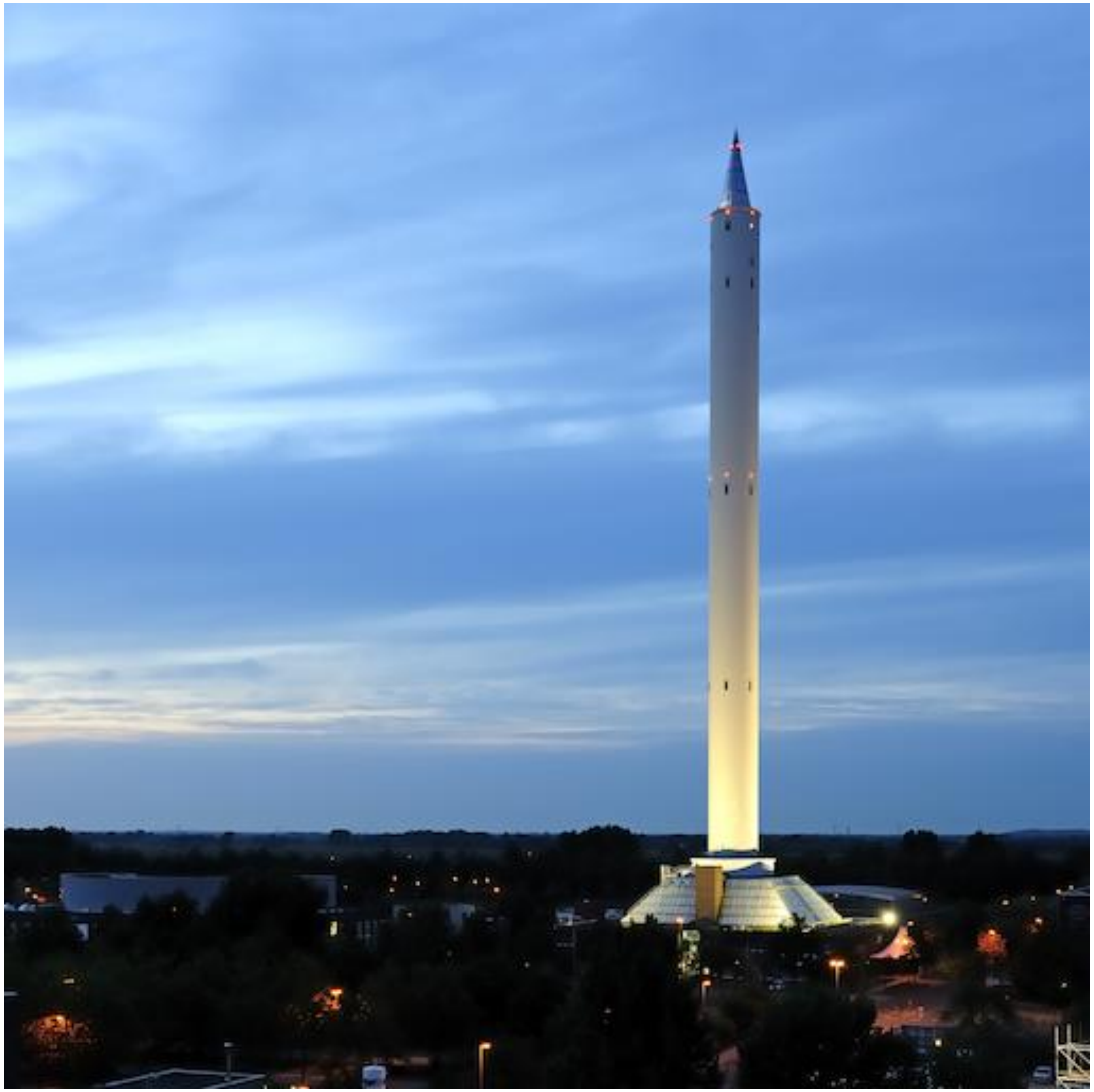
Magnetic levitation



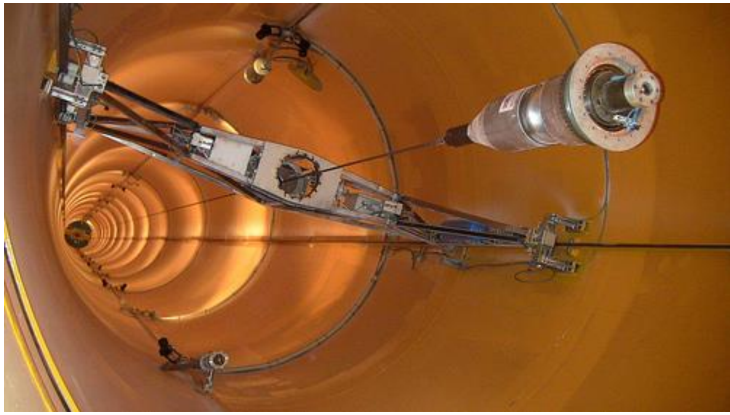


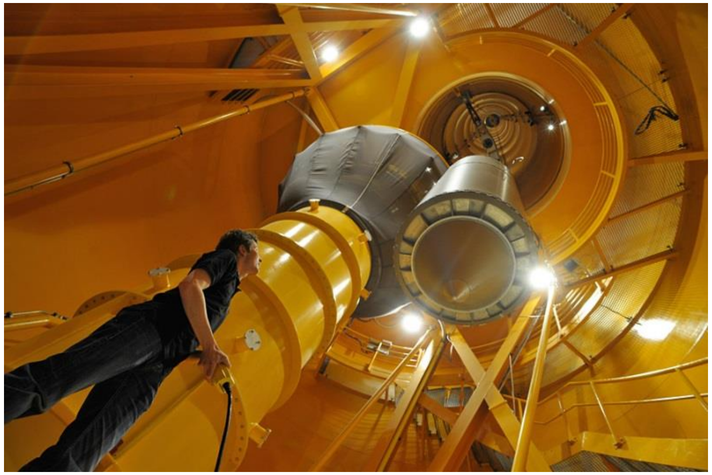
Georg Maret, University of Konstanz, Germany

The Bremen Drop-tower



146 m
9.3 s of weightlessness

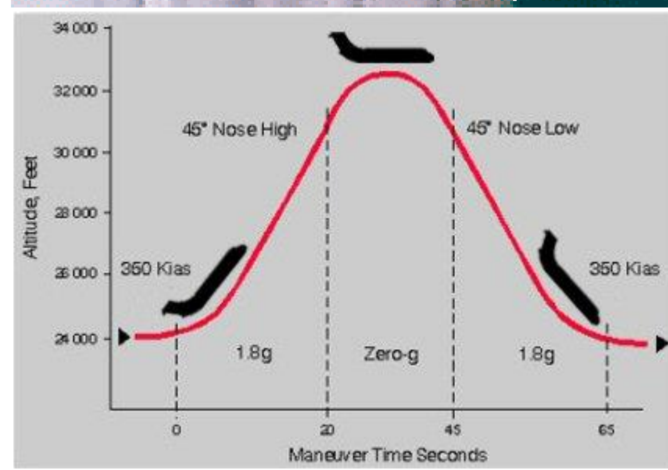




Low gravity environment

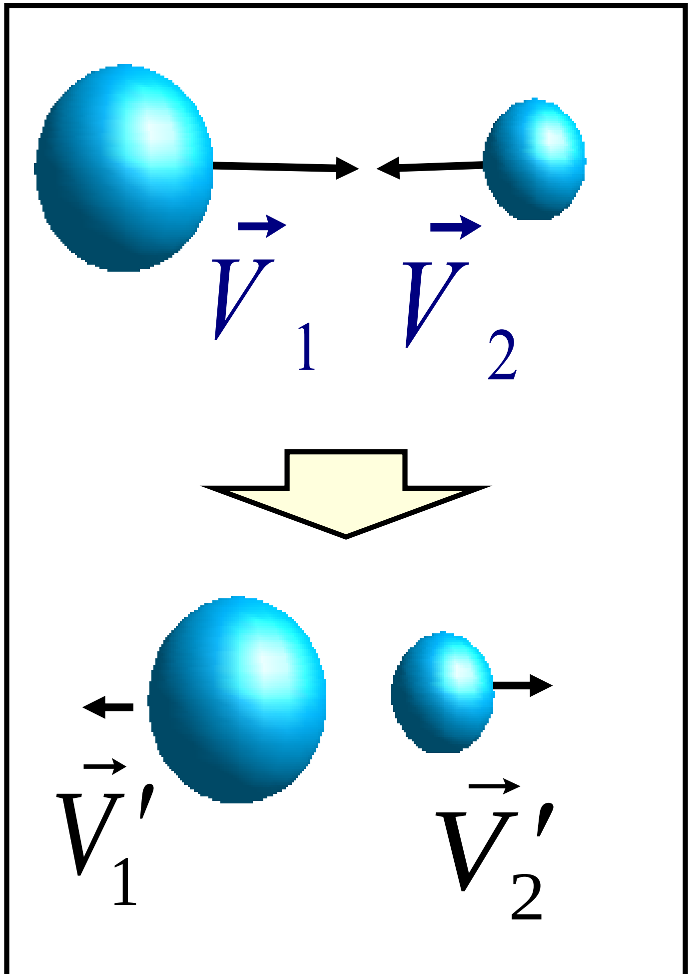
Rocket experiments *(12 min of microgravity)*

Eric Falcon, Univ Paris Diderot,
Sorbonne Paris Cité, Paris, France



Parabolic flights
(30 parabolas,
20 s of reduced gravity)

Restitution coefficient ε



$$\varepsilon = \frac{\left|\vec{V}_1' - \vec{V}_2'\right|}{\left|\vec{V}_1 - \vec{V}_2\right|} = \frac{\left|\vec{V}_{12}'\right|}{\left|\vec{V}_{12}\right|}$$

Energy loss:

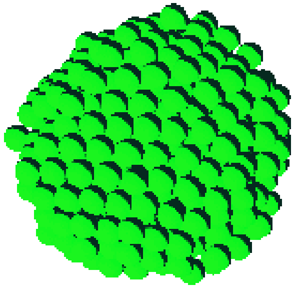
$$\Delta E = -\tfrac{1}{4}\left(1 - \varepsilon^2\right) m V_{12}^2$$

$$0 \le \varepsilon \le 1$$

perfectly inelastic collision

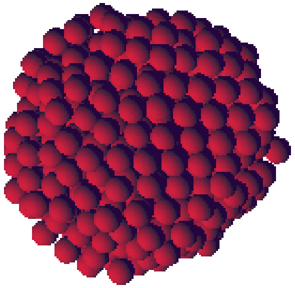
perfectly elastic collision

Restitution coefficient ε



Dependence of *ε* on the relative velocity V_{12}:

$$\boxed{\varepsilon\left(V_{12}\right)=1-C_1V_{12}^{1/5}+C_2V_{12}^{2/5}\mp\ldots}$$

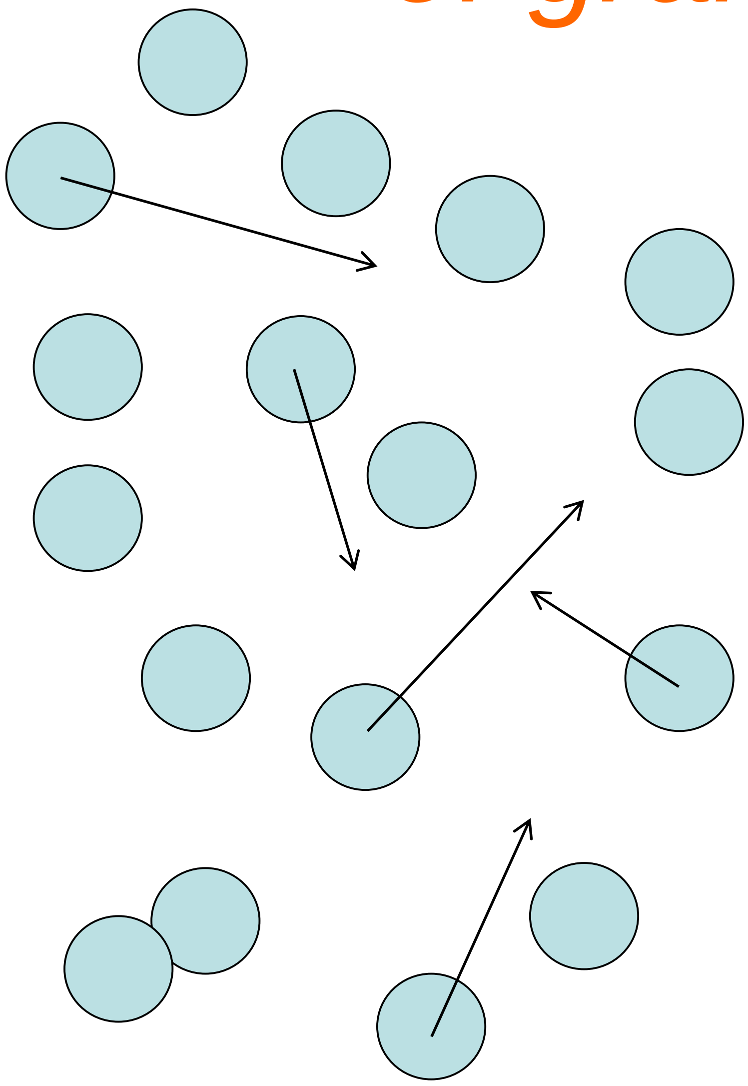


Coefficients C_1, C_2 depend on *Young's modulus, Poisson ratio, viscosity, density* and *sizes* of colliding particles

For *small* collisional relative velocities $\boxed{V_{12}\to 0}$

collisions become *more elastic*: $\boxed{\varepsilon\to 1}$

Event-driven simulations of granular gases



Restitution coefficient

$$\varepsilon = -\frac{\left(\vec{v}\,'_{12}\cdot\vec{n}\right)}{\left(\vec{v}_{12}\cdot\vec{n}\right)}$$

$$\vec{v}\,'_1 = \vec{v}_1 - \frac{1}{2}\left(1+\varepsilon\right)\left(\vec{v}_{12}\cdot\vec{n}\right)\vec{n}$$

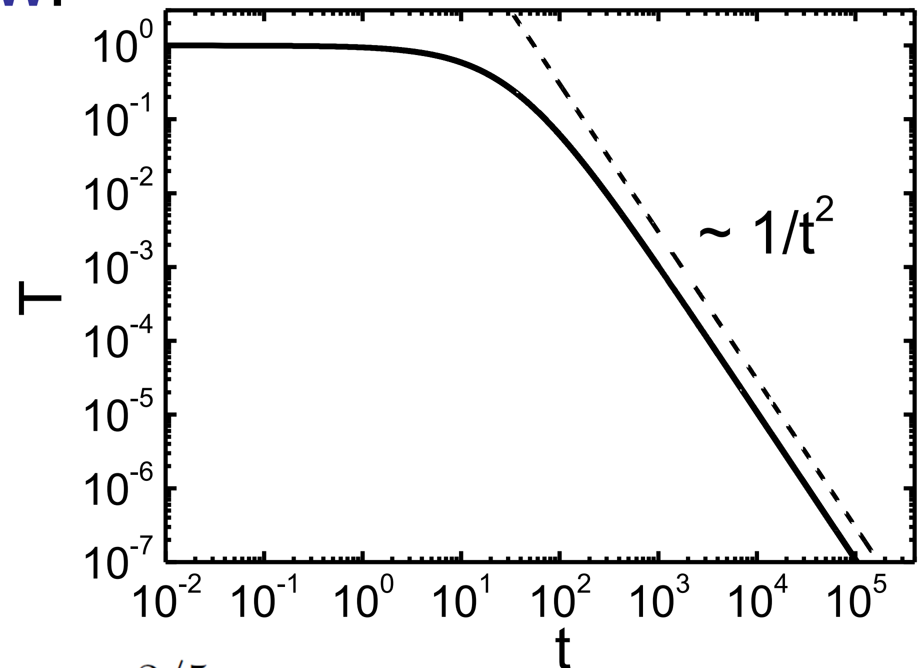
$$\vec{v}\,'_2 = \vec{v}_2 + \frac{1}{2}\left(1+\varepsilon\right)\left(\vec{v}_{12}\cdot\vec{n}\right)\vec{n}$$

Instantaneous binary collisions of particles

Granular temperature

$$T(t) = \frac{2}{3}\int d\vec{v}\,\frac{mv^2}{2}\,f(\vec{\mathbf{v}},\tau)$$

The mean kinetic energy (granular temperature) in a granular gas decreases due to inelastic collisions according to the Haff's law:



$$T(t) = \frac{T_0}{\left(1+t/\tau_0\right)^2}$$

for $\varepsilon = \mathrm{const}$

$$T(t) = \frac{T_0}{\left(1+t/\tau_0\right)^{5/3}}$$

for $\varepsilon(V_{12}) = 1 - C_1 V_{12}^{1/5} + C_2 V_{12}^{2/5} \mp \ldots$

Diffusion coefficient

Diffusion coefficient of granular particles is time-dependent:

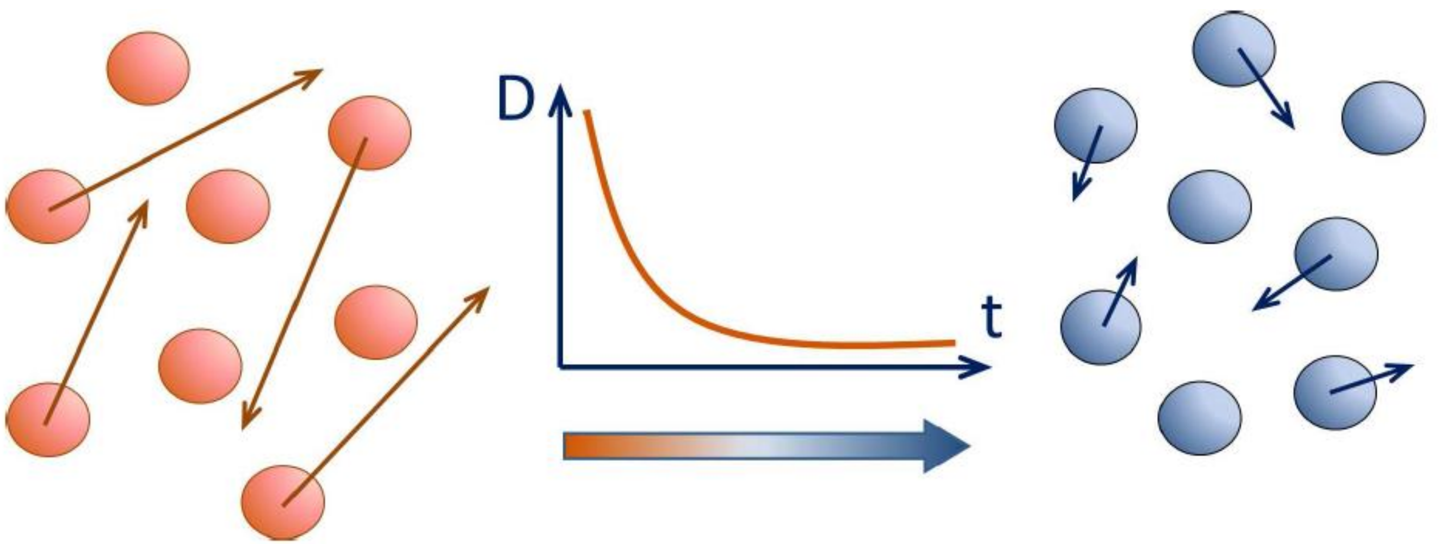
$$D(t) = \frac{d\left\langle r^2(t)\right\rangle}{dt}$$

$$D(t) = \frac{T(t)\tau_v(t)}{m}$$

$\tau_v(t)$ - velocity correlation time

$$\tau_v^{-1}(t) = \frac{2}{3}\sqrt{\pi}\sigma^2 n \sqrt{\frac{T(t)}{m}}(1+\varepsilon)^2$$

$$\boxed{D(t) \sim \sqrt{T(t)}}$$



Diffusion coefficient of rough granular partcles

$$D(t) = \frac{T(t)}{m}\left(\frac{2}{3}\tau_c^{-1}(t)\left(\frac{1+\varepsilon}{2}+\frac{q(1+\beta)}{2(1+q)}\right)-\frac{1}{2}\xi\right)^{-1}$$

Moment of inertia

$$I = \frac{qm\sigma^2}{4}$$

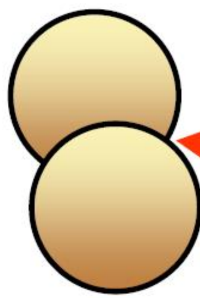
Tangential restitution coefficient

$$\beta = -\frac{g_t'}{g_t}$$

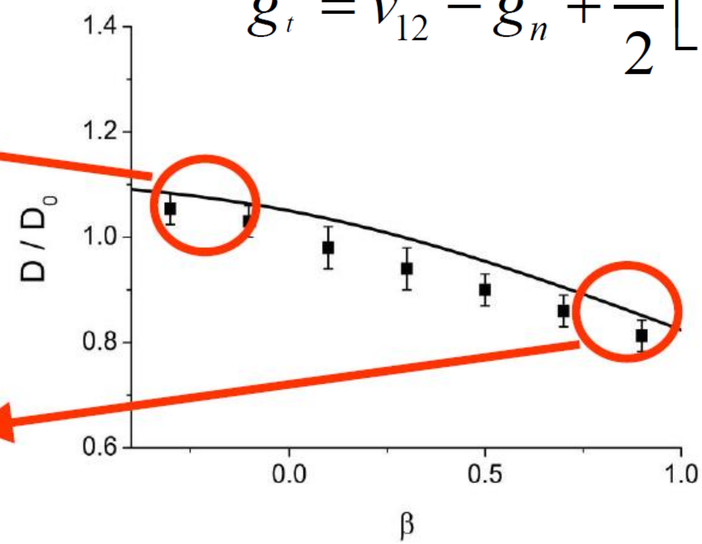
Tangential velocity

$$\vec{g}_t = \vec{v}_{12} - \vec{g}_n + \frac{\sigma}{2}\left[\vec{e}\times\left(\vec{\omega}_1+\vec{\omega}_2\right)\right]$$

Smooth particles



Rough particles



Diffusion coefficient of granular gases

Constant restitution coefficient

$$\varepsilon = const$$

Velocity-dependent restitution coefficient

$$\varepsilon\left(V_{12}\right) = 1 - C_1 V_{12}^{1/5} + C_2 V_{12}^{2/5} \mp \dots$$

Granular temperature
(mean kinetic energy of granular particles)

$$T(t) = T_0 \frac{1}{\left(1 + t/\tau_0\right)^2}$$

$$T(t) = \frac{T_0}{\left(1 + t/\tau_0\right)^{5/3}}$$

Diffusion coefficient

$$D(t) = \frac{T(t)\tau_v(t)}{m} = \frac{D_0}{1 + t/\tau_0}$$

$$D(t) = \frac{D_0}{\left(1 + t/\tau_0\right)^{5/6}}$$

N. V. Brilliantov and T. Poeschel, Phys. Rev. E 61, 1716 (2000)

Overdamped Langevin equation

$$\cancel{m\frac{dv}{dt}} + \gamma(\mathrm{t})v = \sqrt{2D(\mathrm{t})}\gamma(\mathrm{t})\xi(\mathrm{t})$$

White Gaussian noise

$$v = \sqrt{2D(\mathrm{t})}\xi(\mathrm{t})$$

$$\langle \xi(\mathrm{t}_1)\xi(\mathrm{t}_2)\rangle = \delta\left(t_1 - t_2\right)$$

Scaled Brownian motion (SBM)	Ultraslow SBM
Diffusion coefficient	
$D(\mathrm{t}) = \dfrac{\alpha D_0}{t^{1-\alpha}} \quad \alpha > 0$	$D(\mathrm{t}) = \dfrac{D_0}{1+t/\tau_0} \sim \dfrac{1}{t} \quad \alpha = 0$
Mean-squared displacement (MSD)	
$\langle x^2(\mathrm{t})\rangle = 2D_0 t^{\alpha}$	$\langle x^2(\mathrm{t})\rangle = 2D_0\tau_0 \log\left(1+t/\tau 0\right)$

Underdamped Langevin equation with time-dependent temperature

$$\boxed{m\frac{dv}{dt} + \gamma(t)v = \sqrt{2D(t)}\gamma(t)\xi(t)}$$

Friction coefficient

$$\gamma(t) = \frac{m}{\tau_v(0)}\sqrt{\frac{T(t)}{T(0)}}$$

Temperature

$$T(t) = \frac{T(0)}{(1+t/\tau_0)^{2-2\alpha}}$$

Diffusion coefficient

$$D(t) = \frac{D_0}{(1+t/\tau_0)^{1-\alpha}}$$

Mean-squared displacement

$$\left\langle x^2(t)\right\rangle = 2D_0\left[\frac{\tau_0}{\alpha}\left(\left(1+\frac{t}{\tau_0}\right)^{\alpha}-1\right)+\tau_v(0)\left(\exp\left(-\frac{\tau_0}{\alpha\tau_v(0)}\left[\left(1+\frac{t}{\tau_0}\right)^{\alpha}-1\right]\right)-1\right)\right]$$

For large times $t \gg \tau_0$ *SBM result* $\left\langle x^2(t)\right\rangle = \dfrac{2D_0\tau_0^{1-\alpha}}{\alpha}t^{\alpha}$

For small times $t \ll \tau_0$ *ballistic behavior* $\left\langle x^2(t)\right\rangle = \dfrac{D_0 t^2}{\tau_v(0)}$

Time-averaged MSD

$$\left\langle \overline{\delta^2(\Delta)} \right\rangle = \left\langle \overline{\delta_0^2(\Delta)} \right\rangle + \langle \Xi(\Delta) \rangle$$

Underdamped

Overdamped

$$\left\langle \overline{\delta_0^2(\Delta)} \right\rangle = \frac{2D_0\tau_0^2}{\alpha(\alpha+1)(t-\Delta)} \times \left[1 + \left(1+\frac{t}{\tau_0}\right)^{\alpha+1} - \left(1+\frac{\Delta}{\tau_0}\right)^{\alpha+1} - \left(1+\frac{t-\Delta}{\tau_0}\right)^{\alpha+1}\right]$$

$$\langle \Xi(\Delta) \rangle = \frac{2D_0\tau_v(0)}{t-\Delta} \int_0^{t-\Delta} dt' \left[\exp\left(-\frac{\hat{\tau}}{\alpha}\left[\left(1+\frac{t'+\Delta}{\tau_0}\right)^{\alpha} - \left(1+\frac{t'}{\tau_0}\right)^{\alpha}\right]\right) - 1\right]$$

For large lag times $\Delta \gg \tau_v(0)\,(t/\tau_0)^{1-\alpha}$

the underdamped and overdamped limits become comparable:

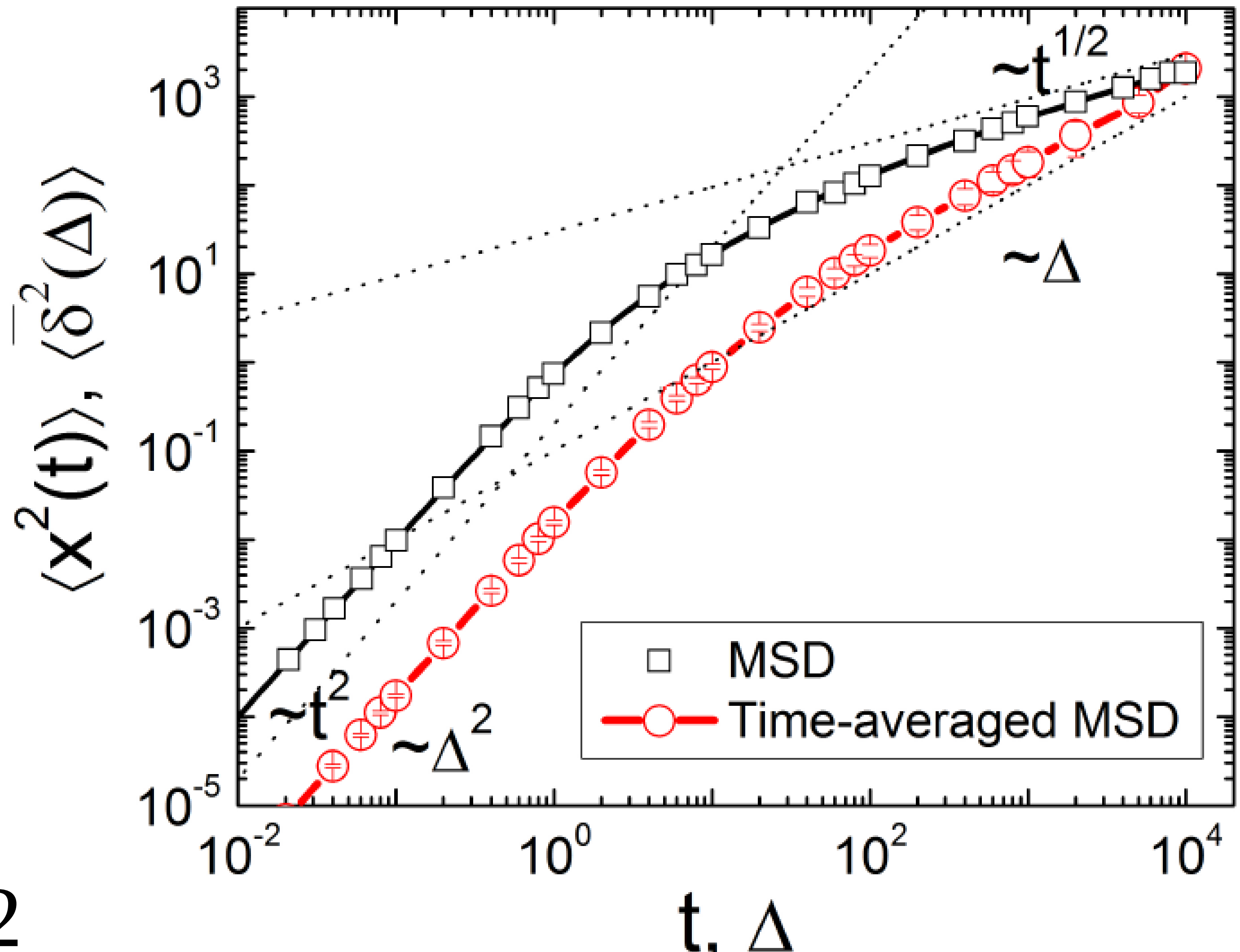
$$\left\langle \overline{\delta^2(\Delta)} \right\rangle \simeq \left\langle \overline{\delta_0^2(\Delta)} \right\rangle \simeq \frac{2D_0\tau_0^{1-\alpha}\Delta}{\alpha t^{1-\alpha}}$$

For intermediate lag times $\Delta \ll \tau_v(0)\,(t/\tau_0)^{1-\alpha} \ll t$

$$\left\langle \overline{\delta^2(\Delta)} \right\rangle \simeq \frac{2D_0\tau_0^{1-\alpha}\Delta^{\alpha+1}}{\alpha t}$$

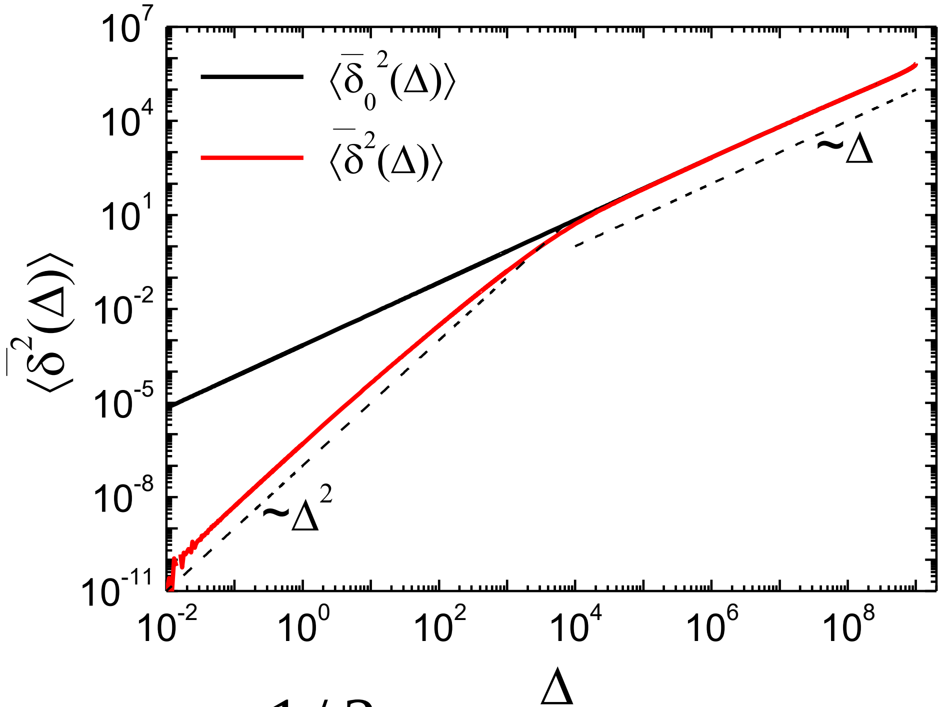
Underdamped SBM

MSD and time-averaged MSD



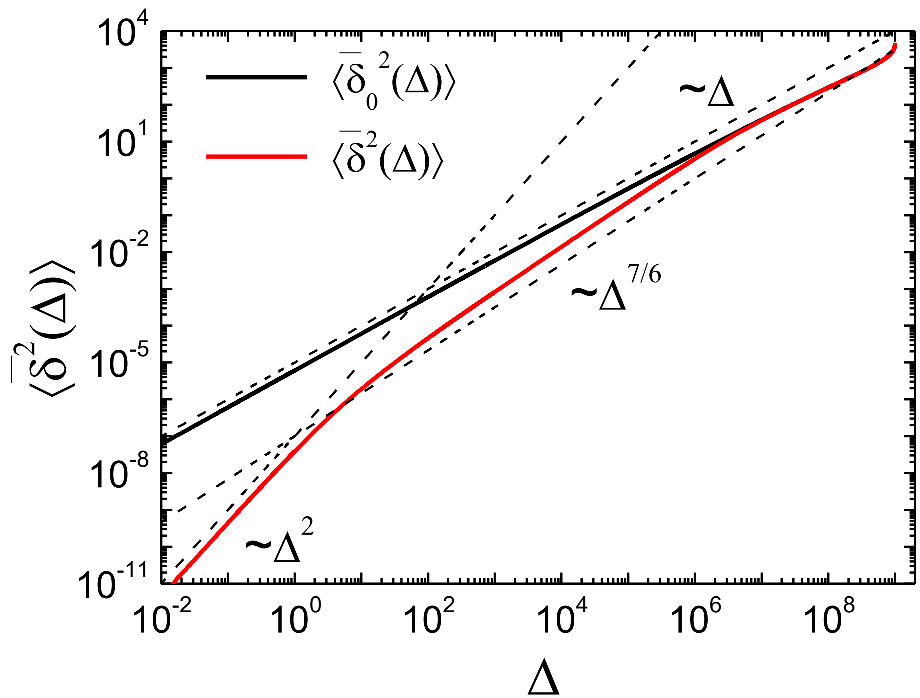
$\alpha = 1/2$

Time-averaged MSD: overdamped and underdamped limits



$\alpha = 1/6$

$$\left\langle \overline{\delta^2(\Delta)} \right\rangle \simeq \frac{2D_0\tau_0^{1-\alpha}\Delta^{\alpha+1}}{\alpha t}$$



$\alpha = 1/2$

$$\left\langle \overline{\delta^2(\Delta)} \right\rangle \simeq \left\langle \overline{\delta_0^2(\Delta)} \right\rangle \simeq \frac{2D_0\tau_0^{1-\alpha}\Delta}{\alpha t^{1-\alpha}}$$

Underdamped Overdamped

Underdamped Langevin equation: ultraslow limit

$$\boxed{\frac{dv}{dt} + \frac{\tau_v^{-1}(0)}{(1+t/\tau_0)}v = \sqrt{\frac{2D_0}{1+t/\tau_0}}\frac{\tau_v^{-1}(0)}{(1+t/\tau_0)}\xi(t)}$$

Friction coefficient

$$\gamma(t) = \frac{m}{\tau_v(0)}\sqrt{\frac{T(t)}{T(0)}}$$

Temperature

$$T(t) = T_0\frac{1}{(1+t/\tau_0)^2}$$

Diffusion coefficient

$$D(t) = \frac{D_0}{1+t/\tau_0}$$

Mean-squared displacement

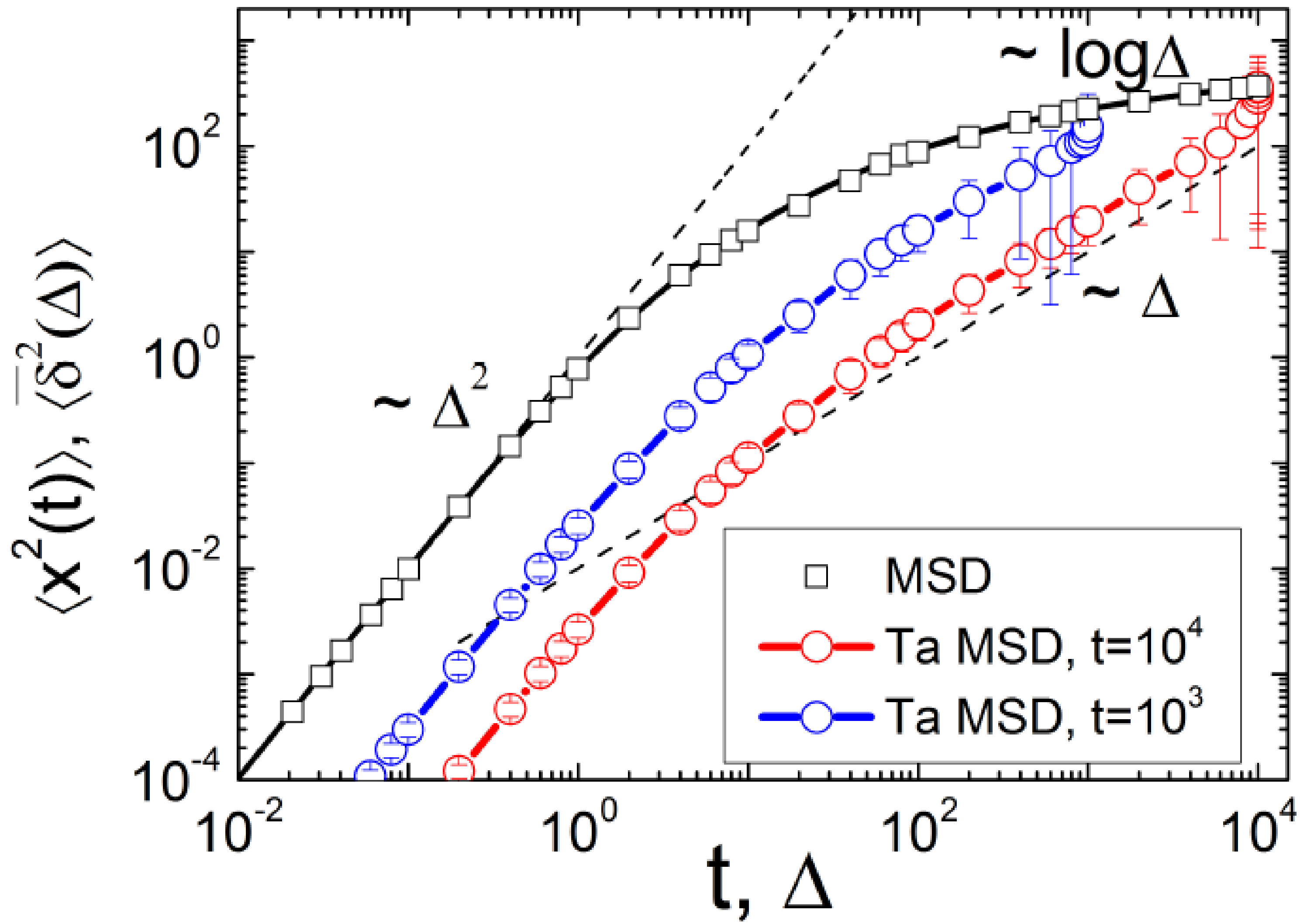
$$\langle x^2(t)\rangle = \frac{2D_0\tau_0^3}{\tau_v^2(0)\left(\hat{\tau}-1\right)^2}\left[\log\left(1+\frac{t}{\tau_0}\right) + \frac{1}{\hat{\tau}-1}\left(\left(1+\frac{t}{\tau_0}\right)^{1-\hat{\tau}} - 1\right)\right]$$

Time-averaged mean-squared displacement

$$\left\langle\overline{\delta^2(\Delta)}\right\rangle \sim \frac{D_0\Delta}{t} \quad \text{for} \quad \tau_0 \ll \Delta \ll t \qquad \hat{\tau} = \frac{\tau_0}{\tau_v(0)}$$

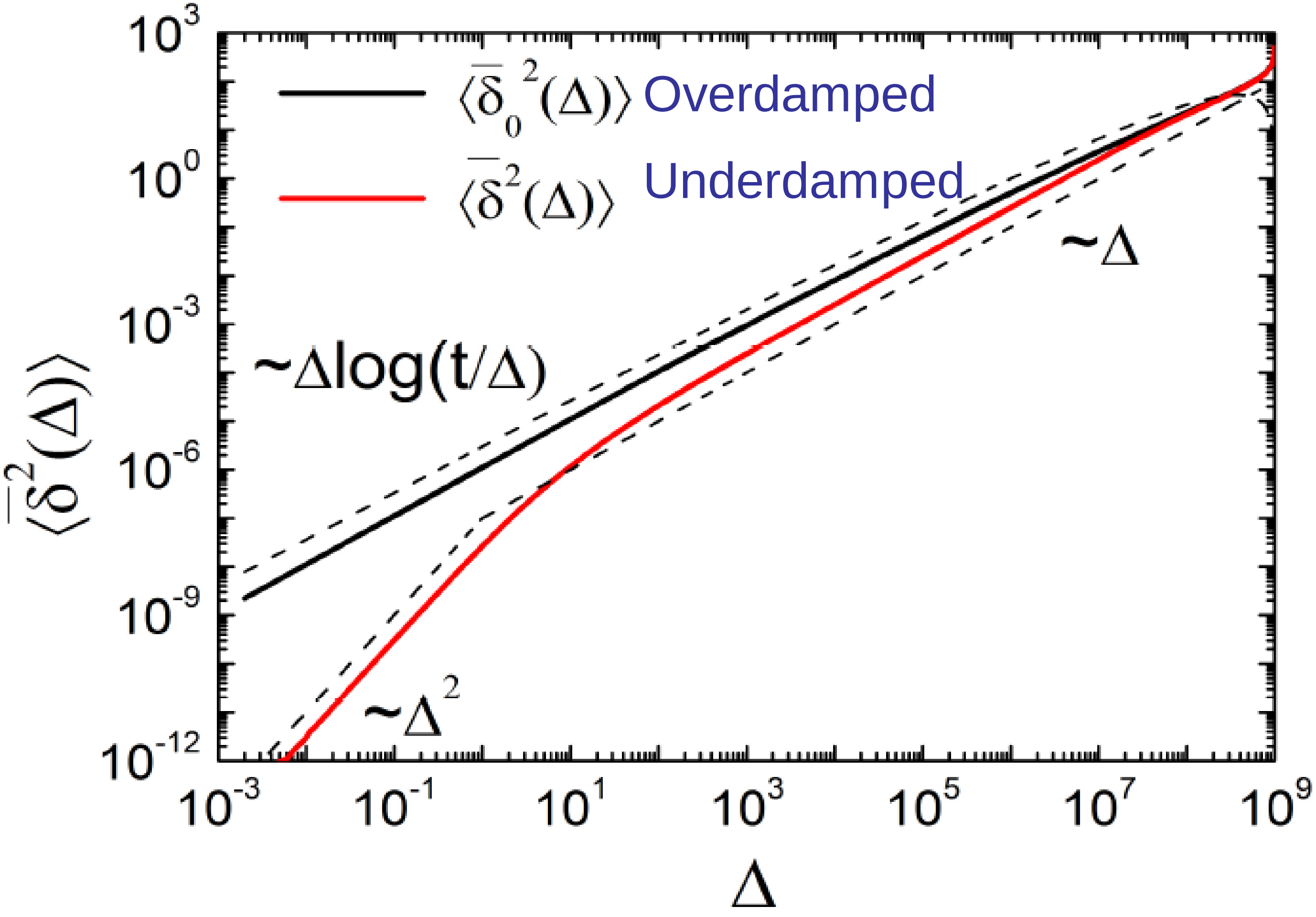
Ultraslow underdamped SBM

MSD and time-averaged MSD



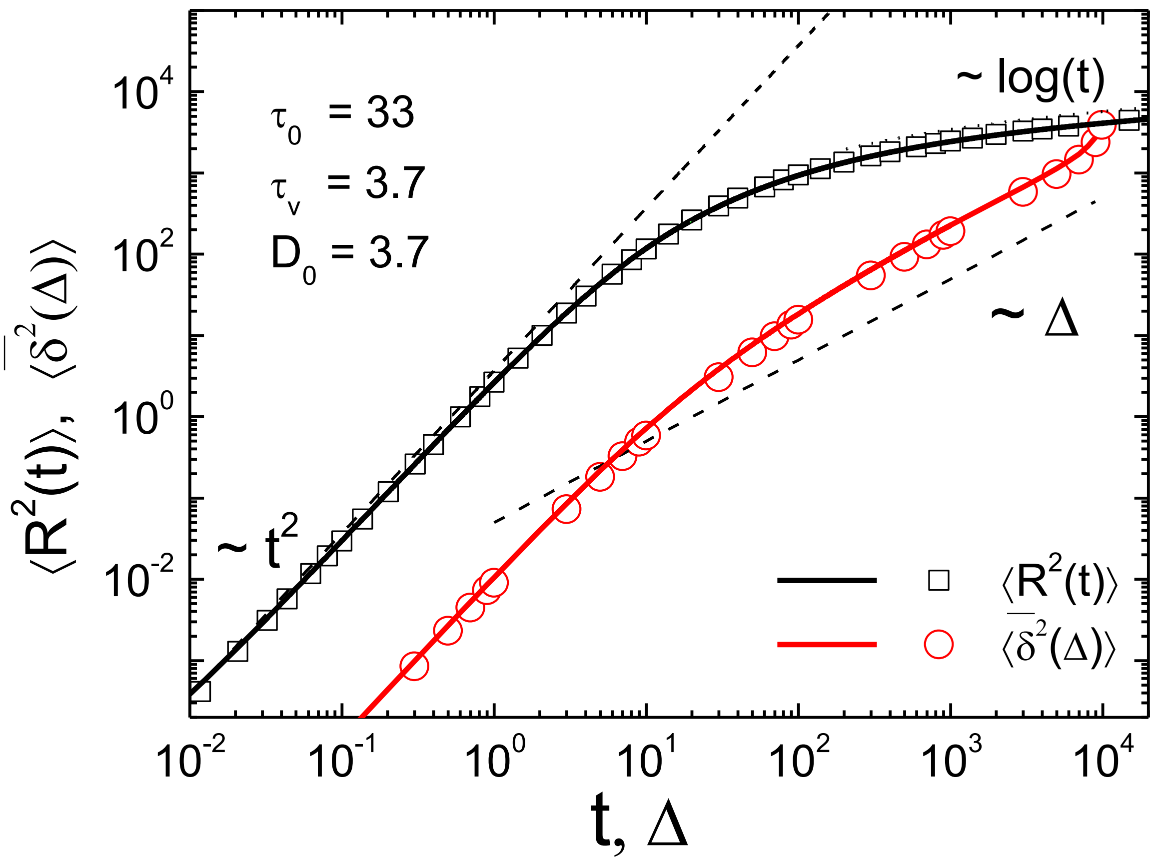
Ultraslow underdamped SBM

Time-averaged MSD: overdamped and underdamped limits



Granular gas with constant restitution coefficient

$$\boxed{\varepsilon(v_{12}) = const}$$



Mean-squared displacement (MSD):

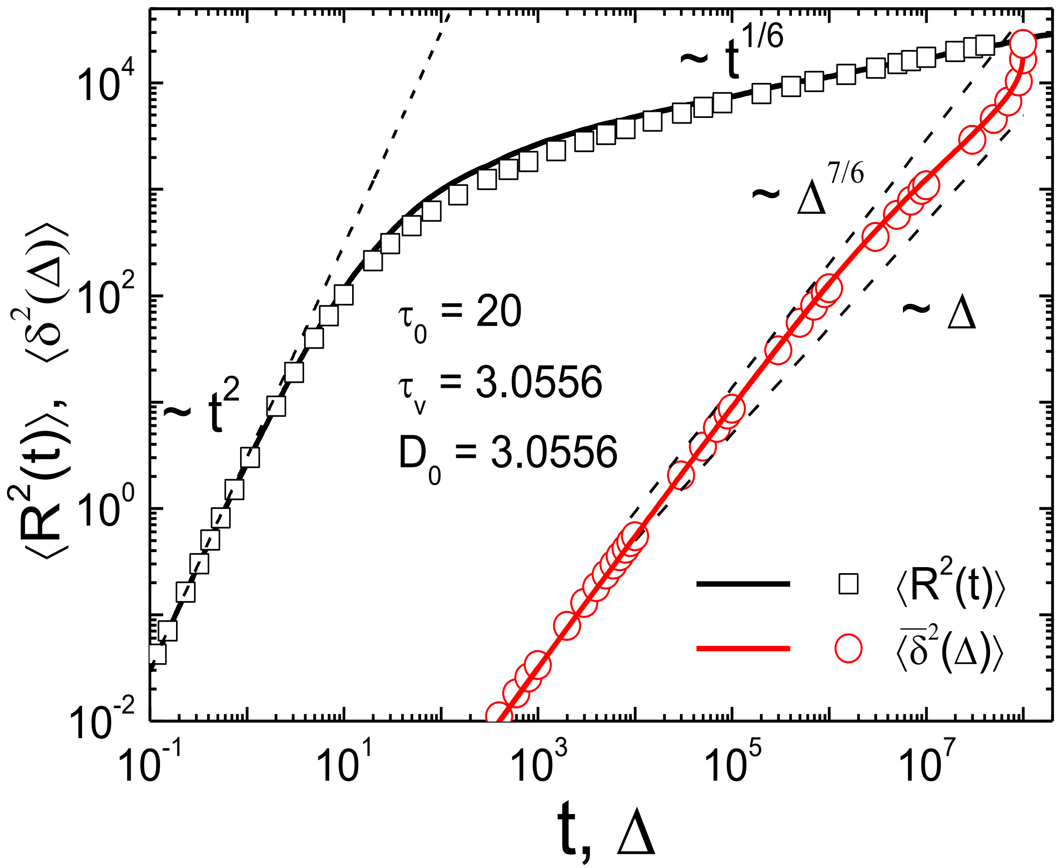
$$\langle R^2(t)\rangle \sim \log t$$

Time-averaged MSD:

$$\left\langle \overline{\delta^2(\Delta)} \right\rangle \sim \Delta/t$$

Granular gas with velocity-dependent restitution coefficient

$$\boxed{\varepsilon(v_{12}) = 1 - C_1 v_{12}^{1/5} + C_2 v_{12}^{2/5} \mp \ldots}$$



Mean-squared displacement (MSD):

$$\langle R^2(t)\rangle \sim t^{1/6}$$

Time-averaged MSD:

Crossover from

$$\langle\delta^2(\Delta)\rangle \sim \Delta^{7/6}/t$$

to $\langle\delta^2(\Delta)\rangle \sim \Delta/t^{5/6}$

Resettnig

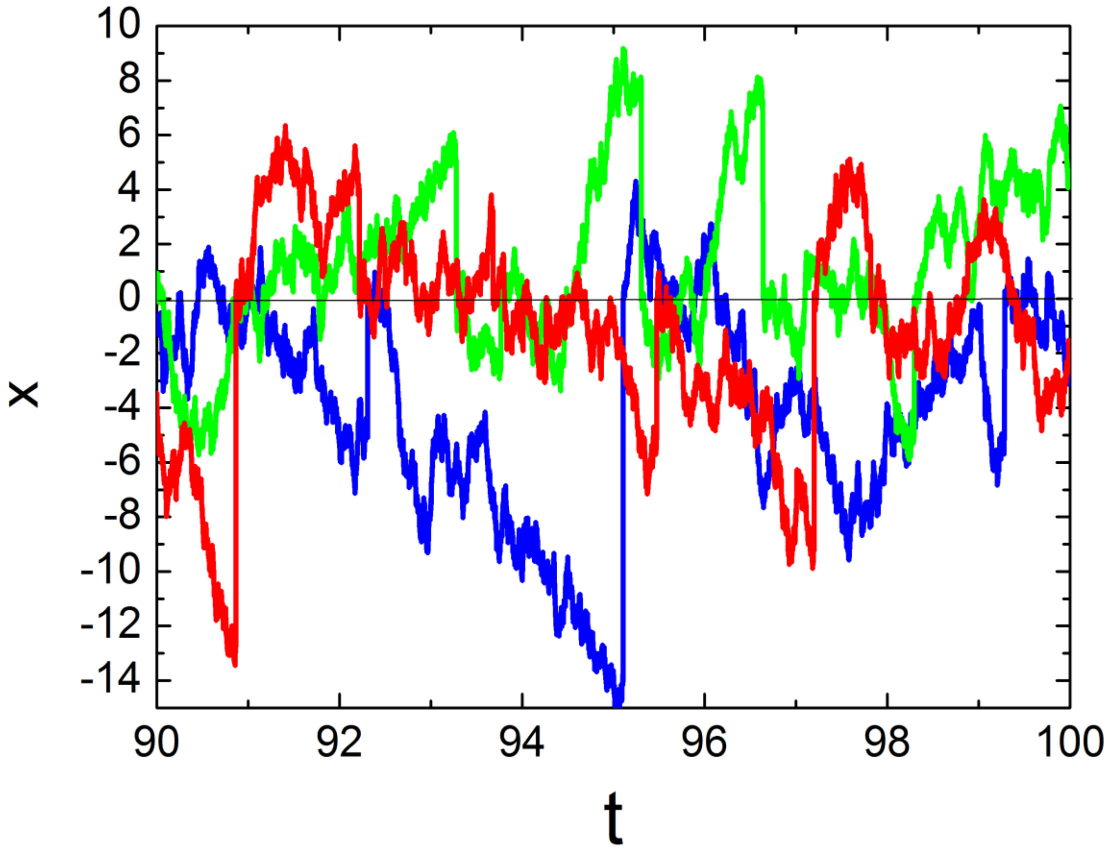
If one searches for a goal and is lost, sometimes it is useful to return to starting point and to start the search from the beginning.

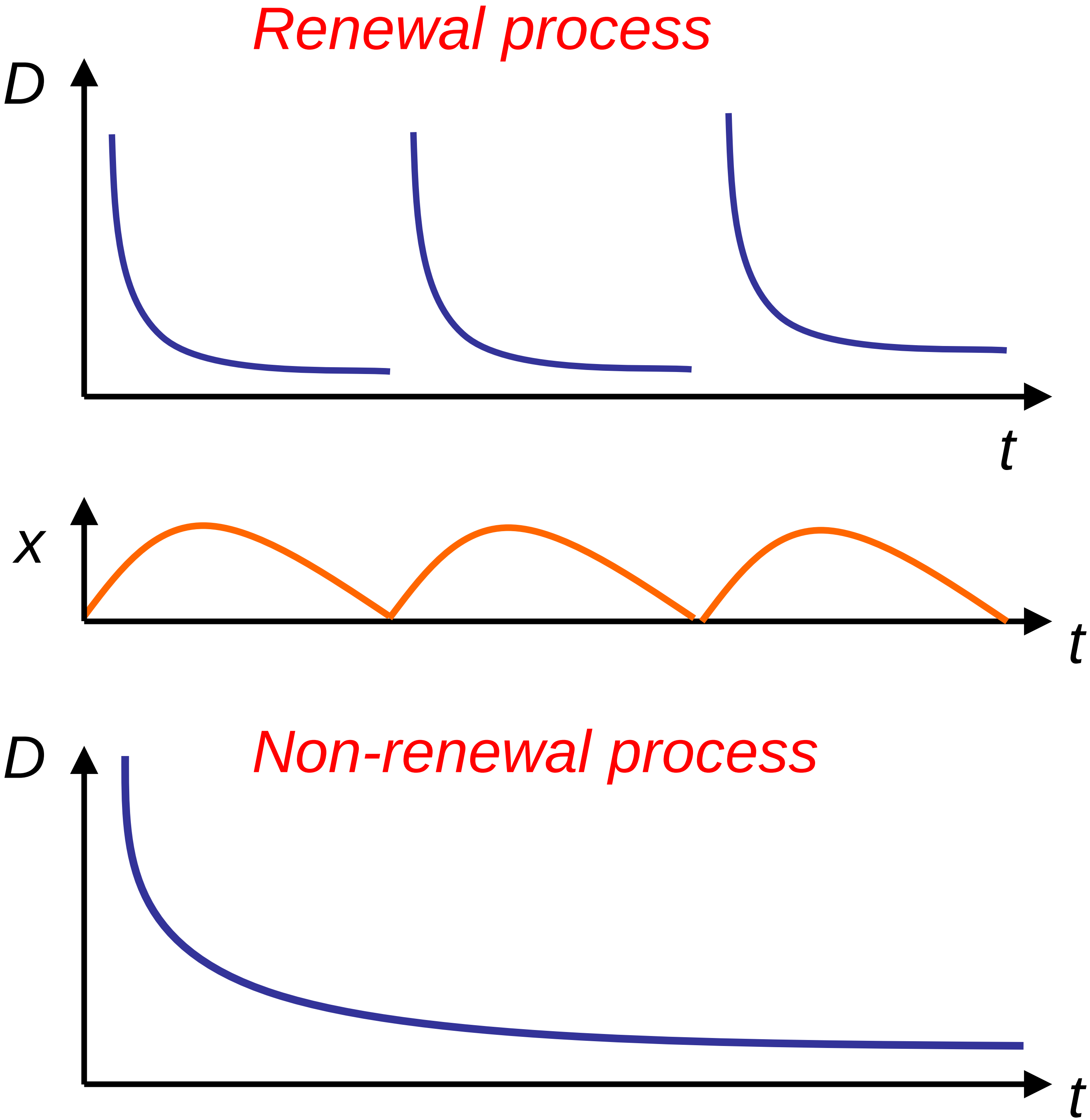
Resetting

- Foraging animals

- Optimizing search algorithms

Typical trajectories for scaled Brownian motion with resetting




Renewal process
D
t
x
t
Non-renewal process
D
t

Scaled Brownian motion

$$\frac{dx(t)}{dt} = \sqrt{2D(t)}\eta(t)$$

White Gaussian noise $\langle \eta(t_1)\eta(t_2) \rangle = \delta\left(t_1 - t_2\right)$

Diffusion coefficient $D(t) = \alpha K_\alpha t^{\alpha - 1}$

Mean-squared displacement $\left\langle x^2(t) \right\rangle = 2K_\alpha t^\alpha$

$\alpha > 1$ Superdiffusion $0 < \alpha < 1$ Subdiffusion

$\alpha = 1$ Ordinary Brownian motion

Probability distribution function

$$p_0\left(x, t\right) = \frac{1}{\sqrt{4\pi K_\alpha t^\alpha}} \exp\left(-\frac{x^2}{4K_\alpha t^\alpha}\right)$$

Distribution of waiting times between the resetting events

Exponential (Poissonian) resetting

$$\psi(t) = re^{-rt}$$

Power-law resetting

$$\psi(t) = \frac{\beta/\tau_0}{(1+t/\tau_0)^{1+\beta}}$$

Survival probability

$$\Psi(t) = e^{-rt}$$

$$\Psi(t) = (1+t/\tau_0)^{-\beta}$$

The probability that an event occurs at time t

$$\phi(t) = r$$

constant rate

$$\beta > 1$$

$$\phi(t) = \frac{\beta - 1}{\tau_0}$$

$$0 < \beta < 1$$

$$\phi(t) = \frac{t^{\beta-1}\tau_0^{-\beta}}{\Gamma(\beta)\Gamma(1-\beta)}$$

Probability distribution function

The probability to find the particle at location x at time t:

$$p(x,t) = \Psi(t)p_0(x,t,0) + \int_0^t dt'\phi(t')\Psi(t-t')p_0(x,t,t')$$

No resetting

Last resetting at time *t'*

Mean-squared displacement

$$\langle x^2(t)\rangle = 2K_\alpha t^\alpha \Psi(t) + 2K_\alpha \int_0^t dt'\phi(t')\Psi(t-t')\left(t^\alpha - t'^\alpha\right)$$

Renewal process: diffusion coefficient also resets to initial value D_0

Non- renewal process: diffusion coefficient is not affected by the resetting events

Mean-squared displacement

Exponential resetting

$$\psi(t) = re^{-rt}$$

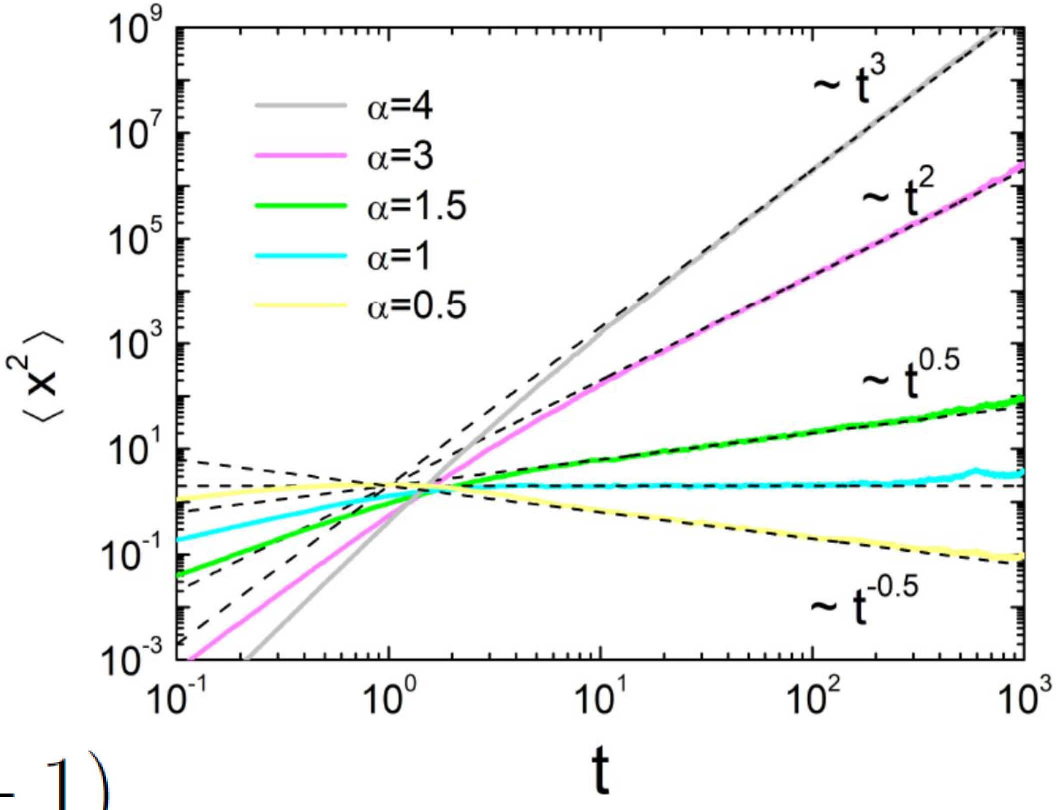
Non-renewal process:

$$\left\langle x^2(t) \right\rangle = \frac{2\alpha K_\alpha}{r} t^{\alpha-1}$$

Renewal process:

Steady state:

$$\left\langle x^2(t) \right\rangle = \frac{2K_\alpha}{r^\alpha} \Gamma\left(\alpha+1\right)$$

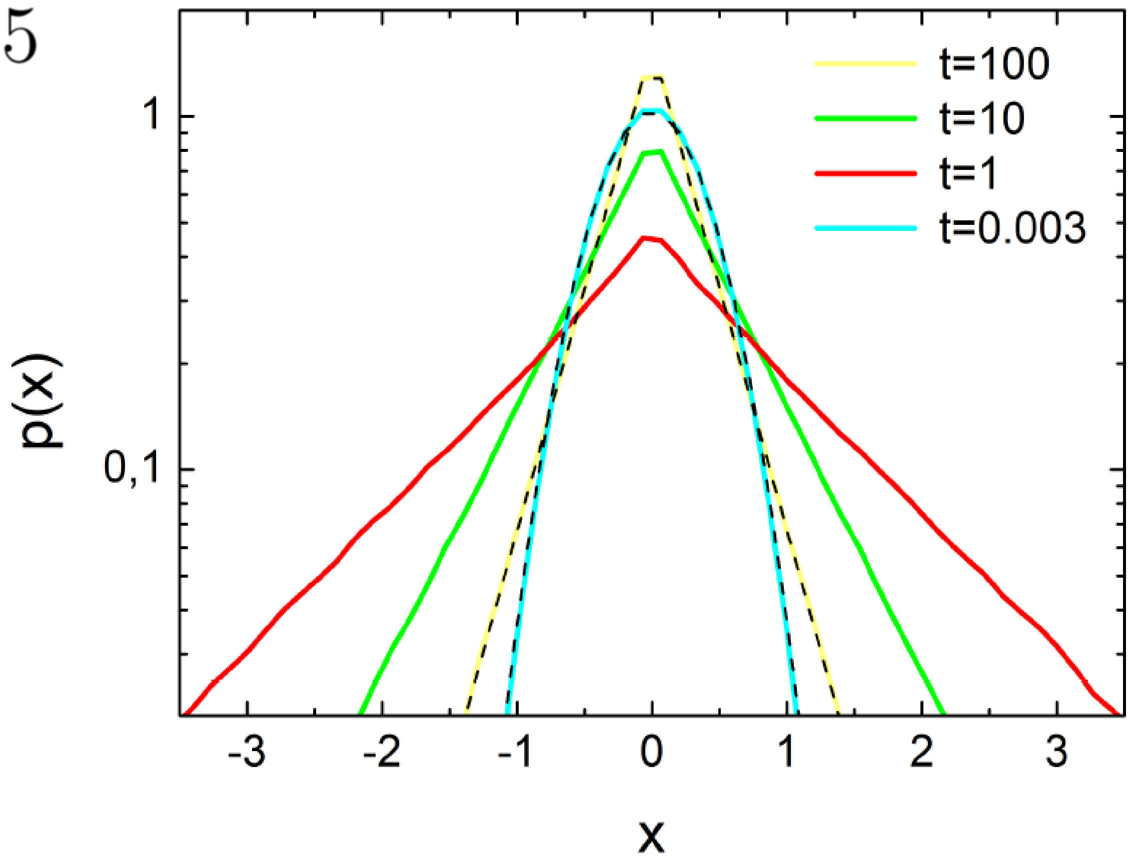


Probability distribution function

Exponential resetting: non-renewal process

$$p(x,t) = \frac{1}{2}\sqrt{\frac{r}{\alpha K_\alpha}} t^{\frac{1-\alpha}{2}} \exp\left(-\sqrt{\frac{r}{\alpha K_\alpha}} |x| \, t^{\frac{1-\alpha}{2}}\right)$$

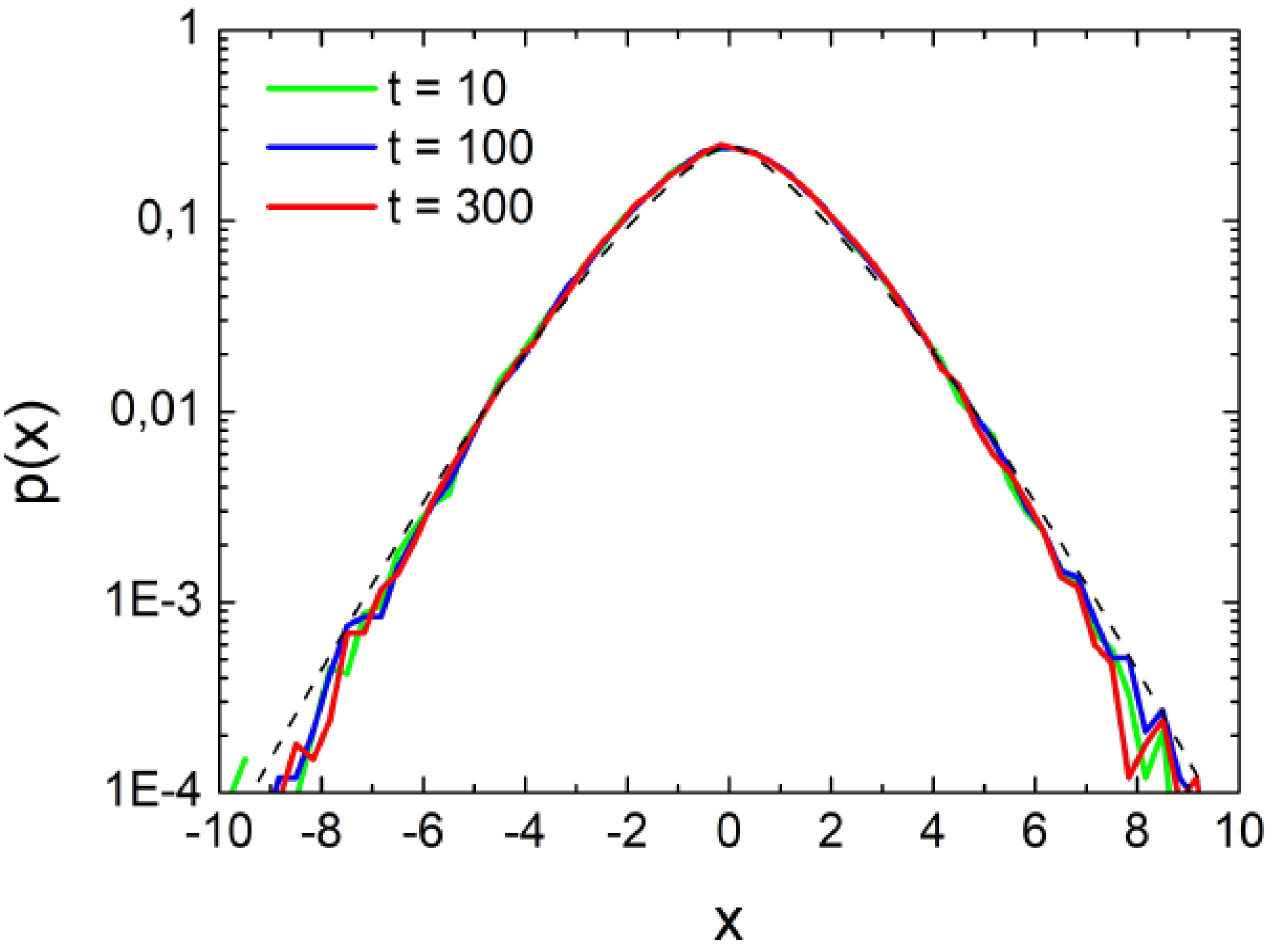
$\alpha = 0.5$



Probability distribution function

Exponential resetting: renewal process

$$p(x,t)=\frac{r\sqrt{2}}{\sqrt{\alpha(\alpha+1)}}\left(\frac{\alpha}{4K_\alpha r}\right)^{\frac{1}{\alpha+1}} x^{\frac{1-\alpha}{2(\alpha+1)}}\exp\left(-\left(\frac{x^2r^\alpha}{4K_\alpha}\right)^{\frac{1}{\alpha+1}}\left(\alpha^{\frac{1}{\alpha+1}}+\alpha^{-\frac{\alpha}{\alpha+1}}\right)\right)$$



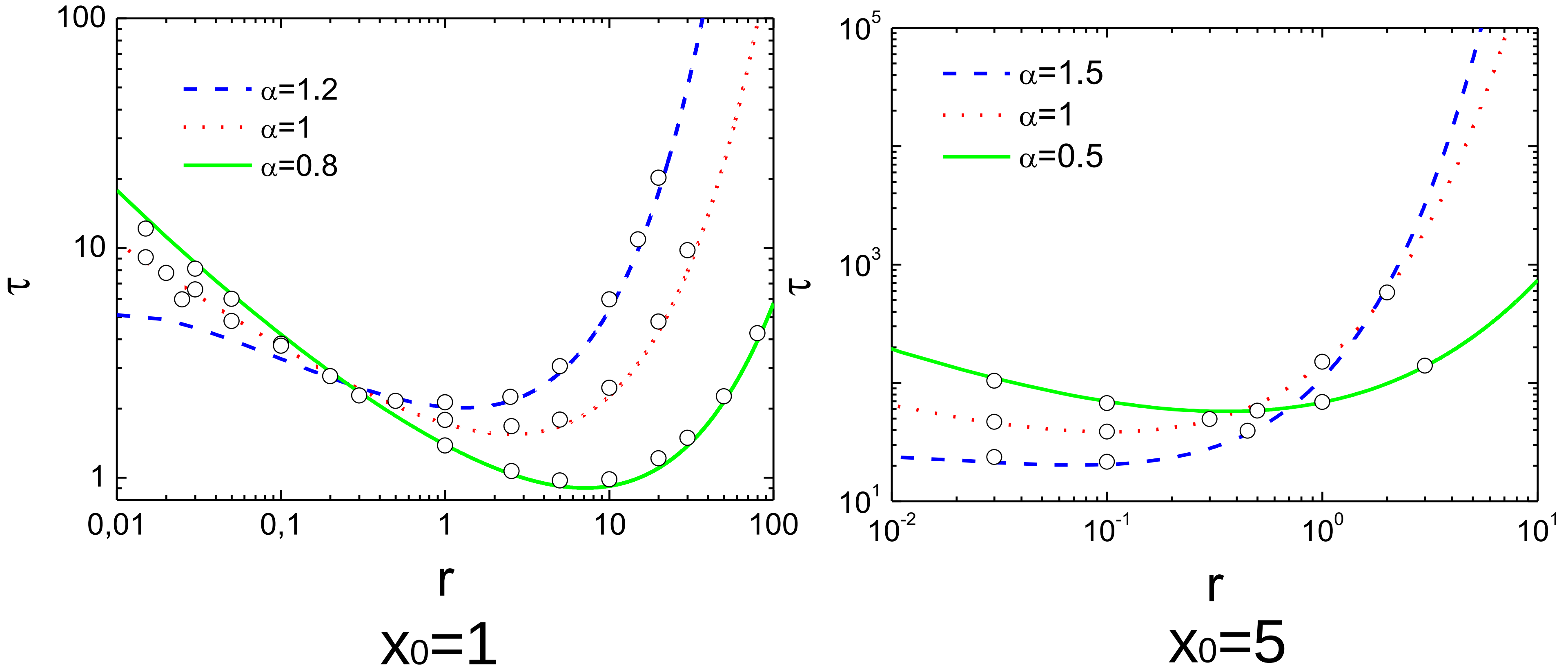
Steady state

$$\alpha = 0.5$$

First passage time

Exponential resetting: renewal process

$$\tau = \frac{1}{r}\left\{\sqrt{\frac{\alpha+1}{2\alpha}}\exp\left[r^{\frac{\alpha}{\alpha+1}}\left(\frac{x_0^2}{4K_\alpha}\right)^{\frac{1}{\alpha+1}}\left(\alpha^{\frac{1}{1+\alpha}}+\alpha^{-\frac{\alpha}{1+\alpha}}\right)\right]-1\right\}$$



Mean-squared displacement

Power-law resetting

non-renewal process

$$\psi(t) = \frac{\beta/\tau_0}{\left(1 + t/\tau_0\right)^{1+\beta}}$$

$0 < \beta < 1$

$$\left\langle x^2(t) \right\rangle = 2K_\alpha t^\alpha \left(1 - \frac{1}{\alpha B\left(\alpha, \beta\right)}\right)$$

$1 < \beta < 2$

$$\left\langle x^2(t) \right\rangle = 2K_\alpha t^{1+\alpha-\beta} \tau_0^{\beta-1} \left(\alpha B\left(\alpha, 2-\beta\right) - 1\right)$$

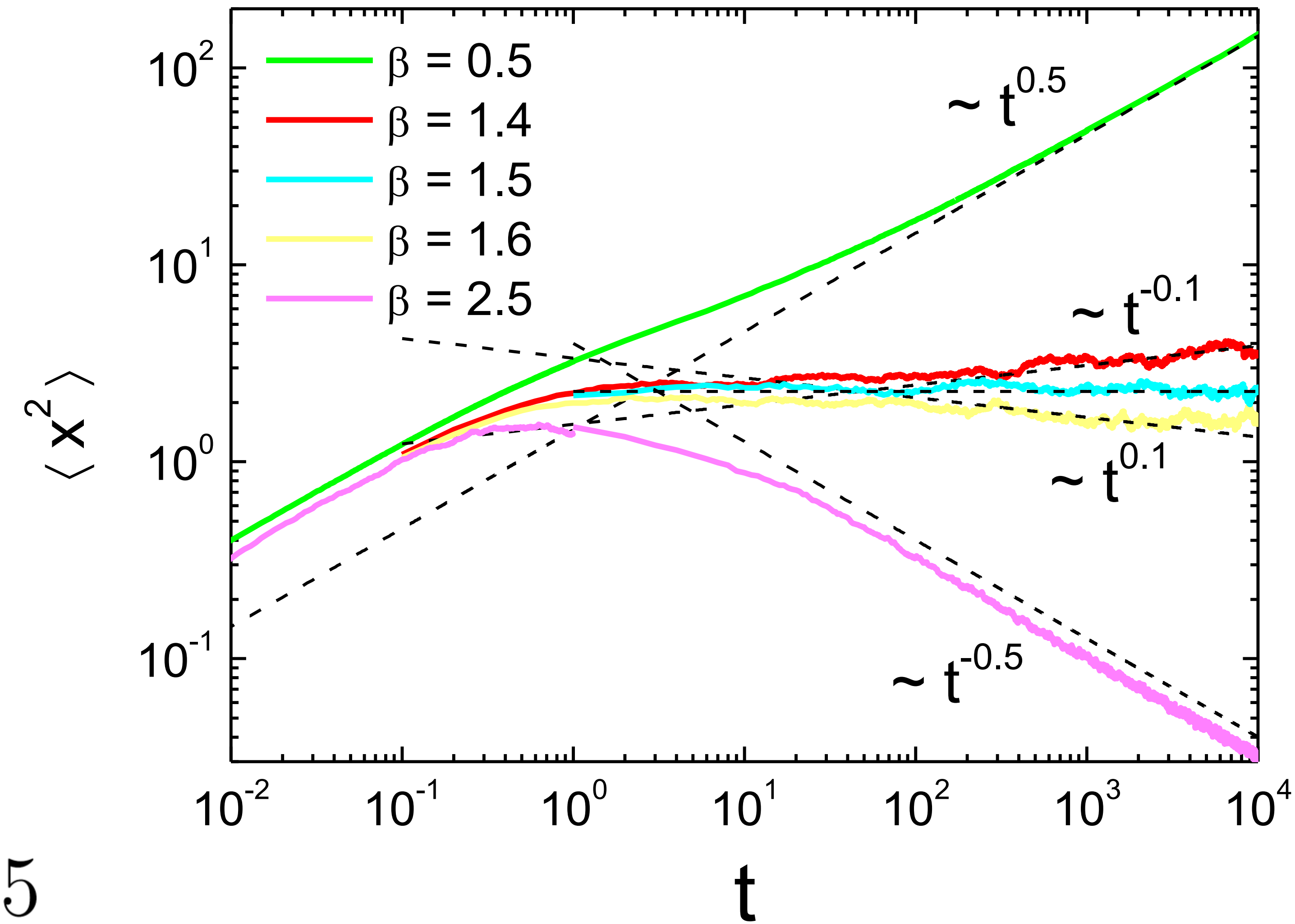
$\beta > 2$

$$\left\langle x^2(t) \right\rangle = \frac{2\alpha K_\alpha \tau_0}{\beta - 2} t^{\alpha-1}$$

Mean-squared displacement

Power-law resetting

non-renewal process



$\alpha = 0.5$

Mean-squared displacement

Power-law resetting: renewal process

$0 < \beta < 1$

$$\left\langle x^2(t)\right\rangle = \frac{2K_\alpha t^\alpha}{\Gamma\left(\beta\right)\Gamma\left(1-\beta\right)}\int_0^1 d\tau\tau^{\beta-1}\left(1-\tau\right)^{\alpha-\beta}$$

$1 < \beta < 2$

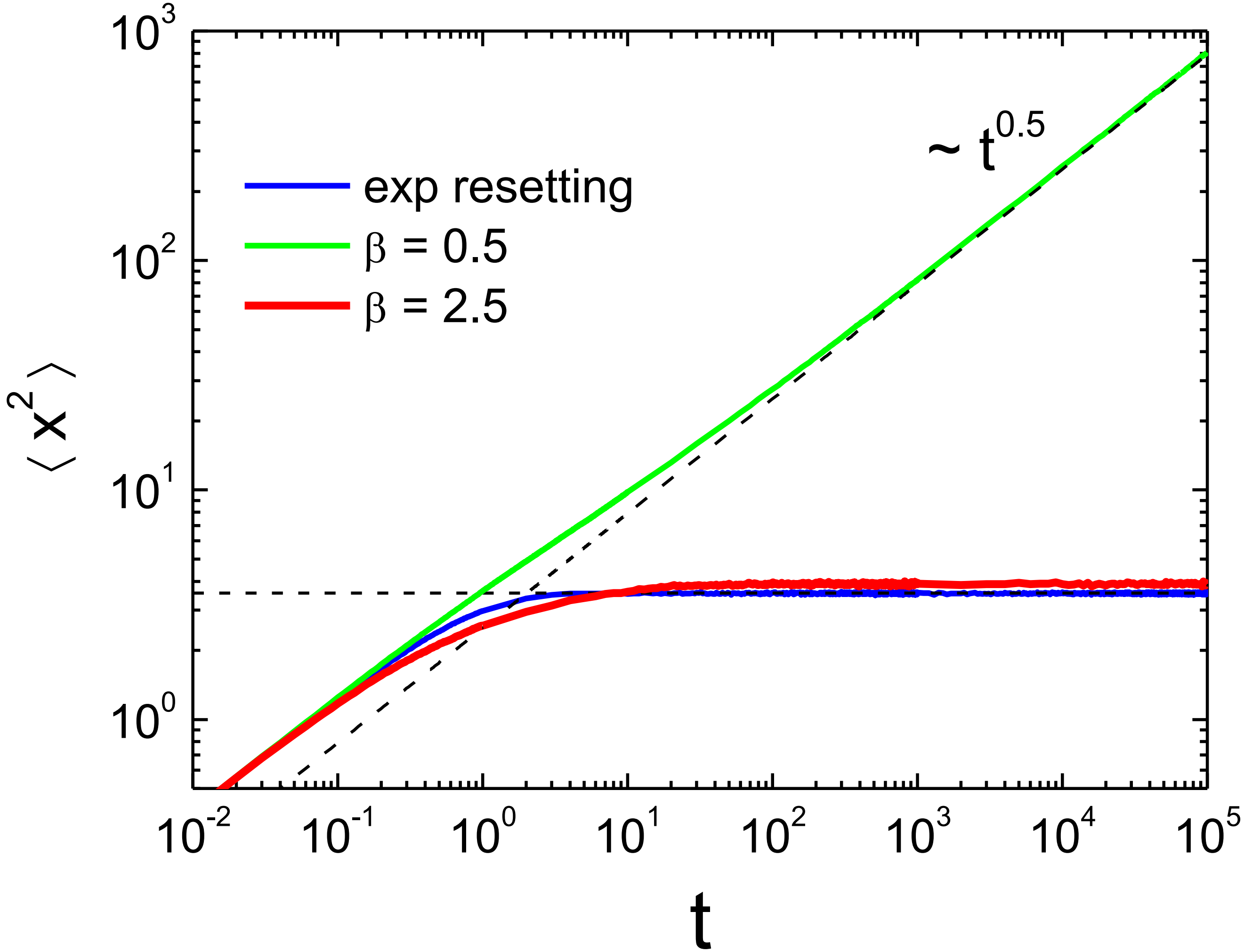
$$\left\langle x^2(t)\right\rangle = \frac{2K_\alpha \tau_0^{\beta-1}\left(\beta-1\right)}{\alpha-\beta+1}t^{\alpha-\beta+1}$$

$\beta > 2$ $\quad \left\langle x^2(t)\right\rangle = \dfrac{\alpha K_\alpha \tau_0}{\beta-2}$

Mean-squared displacement

Power-law resetting

renewal process

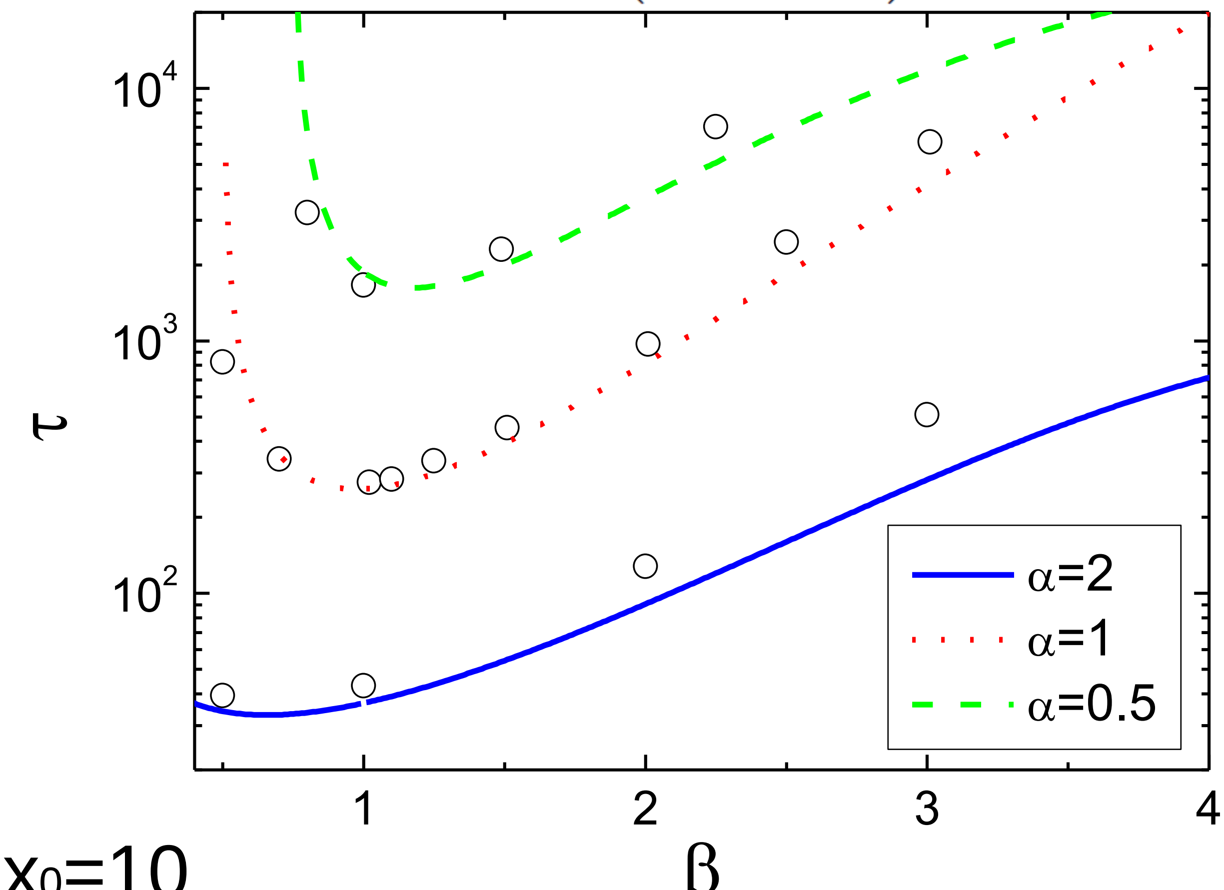


$\alpha = 0.5$

First passage time

Power-law resetting: renewal process

$$\tau \simeq \frac{\tau_0}{\beta - 1} \times \frac{\sqrt{\pi}\mathrm{erf}(\sqrt{\mathrm{A}}) - \mathrm{A}^{1-\beta}\gamma\left(\mathrm{A}, \frac{\beta-1}{\alpha} + \frac{1}{2}\right)}{A^{-\beta}\gamma\left(A, \frac{\beta}{\alpha} + \frac{1}{2}\right) + \sqrt{\pi}\mathrm{erfc}(\sqrt{\mathrm{A}})}$$



x_0=10

Conclusions

- The underdamped scaled Brownian motion, a novel anomalous diffusion process governed by Langevin equation with time-dependent diffusion and friction coefficients, describes the diffusion in bath with time-dependent temperature and in granular gases.
- The overdamped limit of the Langevin equation does not capture all properties of the system
- Resetting significantly affects the behavior of Brownian motion with time-dependent diffusion coefficient

1. Bodrova A., Chechkin A. V., Cherstvy A. G., Metzler R., *Quantifying non-ergodic dynamics of force-free granular gases.* **PCCP** 17, 21791 (2015).
2. Bodrova A., Chechkin A. V., Cherstvy A. G., Metzler R., *Ultraslow scaled Brownian motion.* **New J. Phys**. 17, 063038 (2015)
3 . Bodrova A., Chechkin A. V., Cherstvy A. G., Safdari H., Sokolov I.M., Metzler R., *Underdamped scaled Brownian motion: (non-)existence of the overdamped limit in anomalous diffusion.* **Sci. Rep.** 6**,** 30520 (2016)
4. Bodrova A.S., Chechkin A. V., Sokolov I.M. *Non-renewal resetting of scaled Brownian motion*. Submitted to **Phys. Rev. E** (2018)
5. Bodrova A.S., Chechkin A. V., Sokolov I.M. *Scaled Brownian motion with renewal resetting*. Submitted to **Phys. Rev. E** (2018)