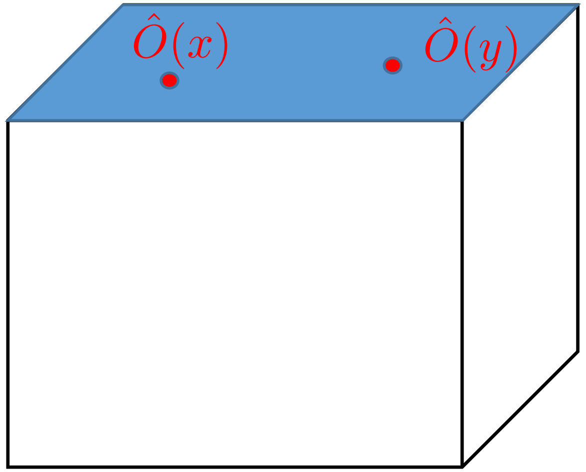
Критический взгляд на критическое поведение поверхности в O(N) модели в d = 3

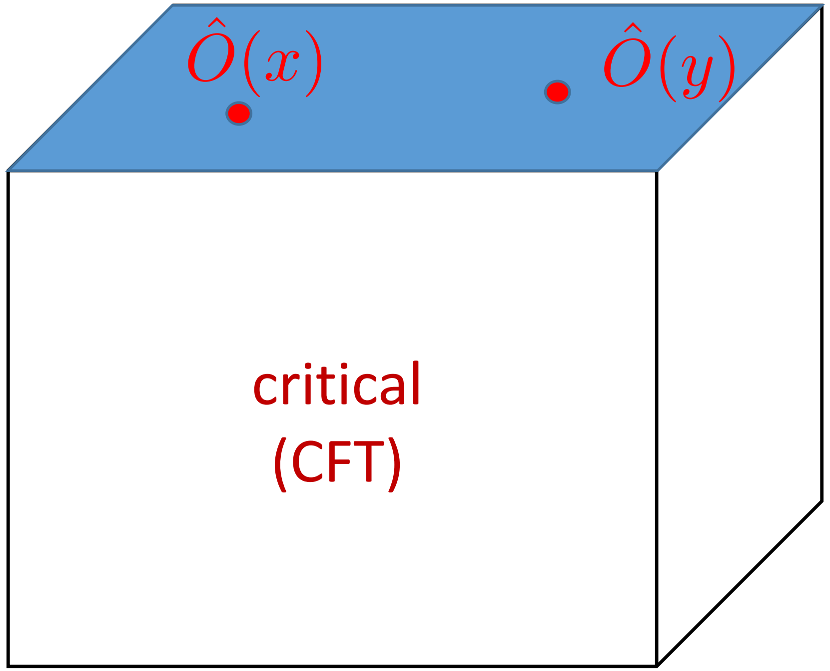
Макс Метлицкий
MIT

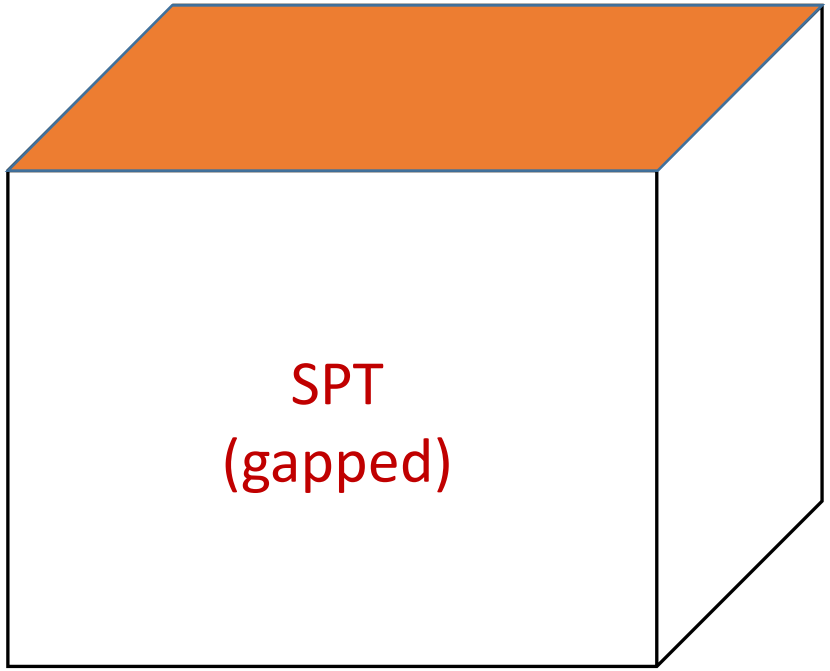
arXiv: 2009.05119



Суперкомпьютерное моделирование в науке и инженерии,
или Вычислительные среды,
ВШЭ, 19 мая, 2021.

Boundary criticality





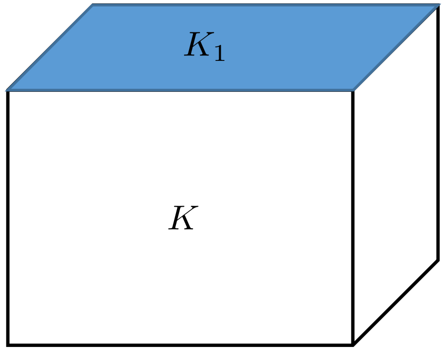
$$\langle \hat{O}(x)\hat{O}(y)\rangle \sim \frac{1}{|x-y|^{2\hat{\Delta}}}$$

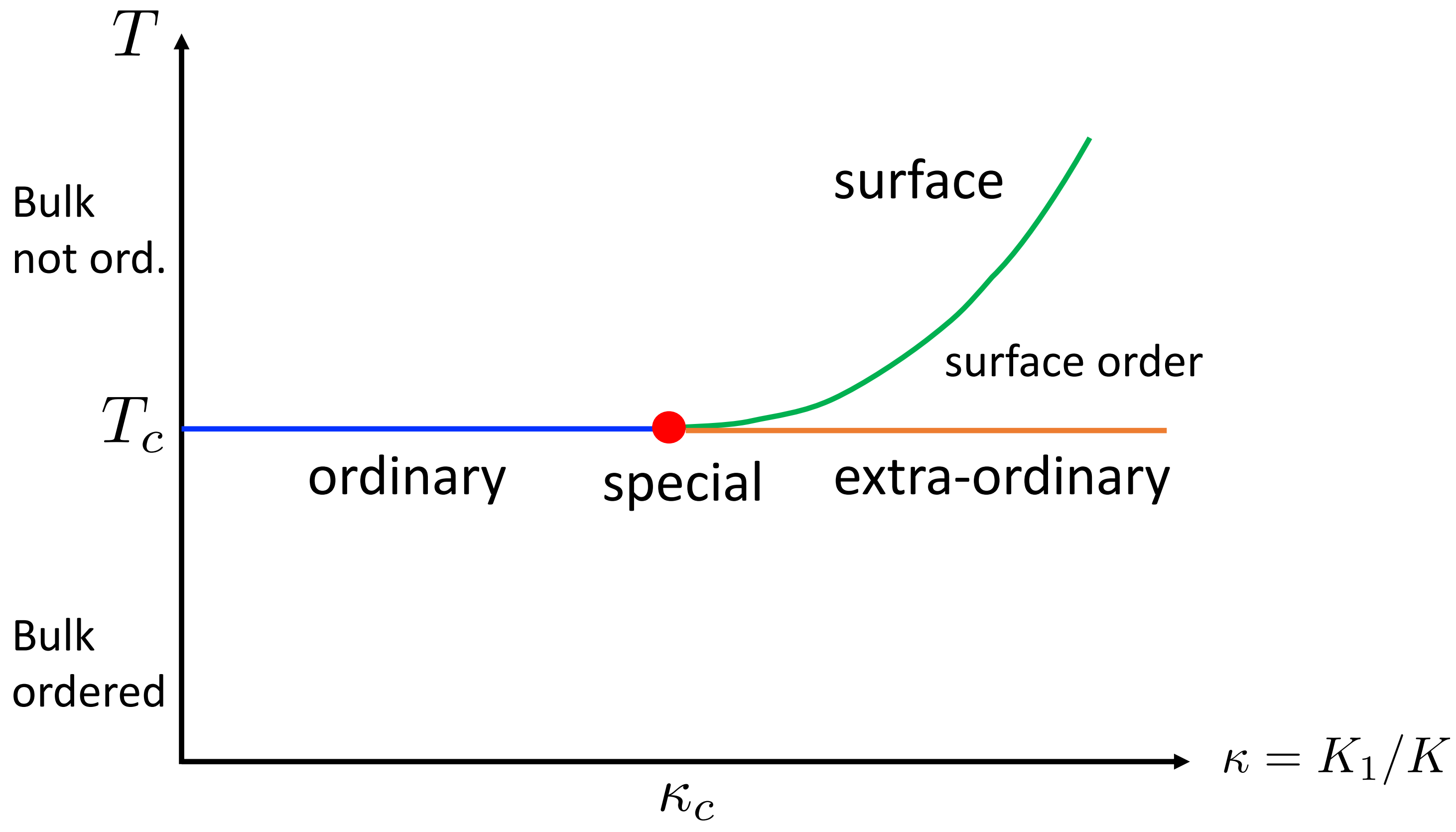
Grover, Vishwanath, 1206.1332 …

- BCFT - not unique

Classical O(N) model

$$H = -\sum_{\langle ij \rangle} K_{ij} \vec{S}_i \cdot \vec{S}_j$$





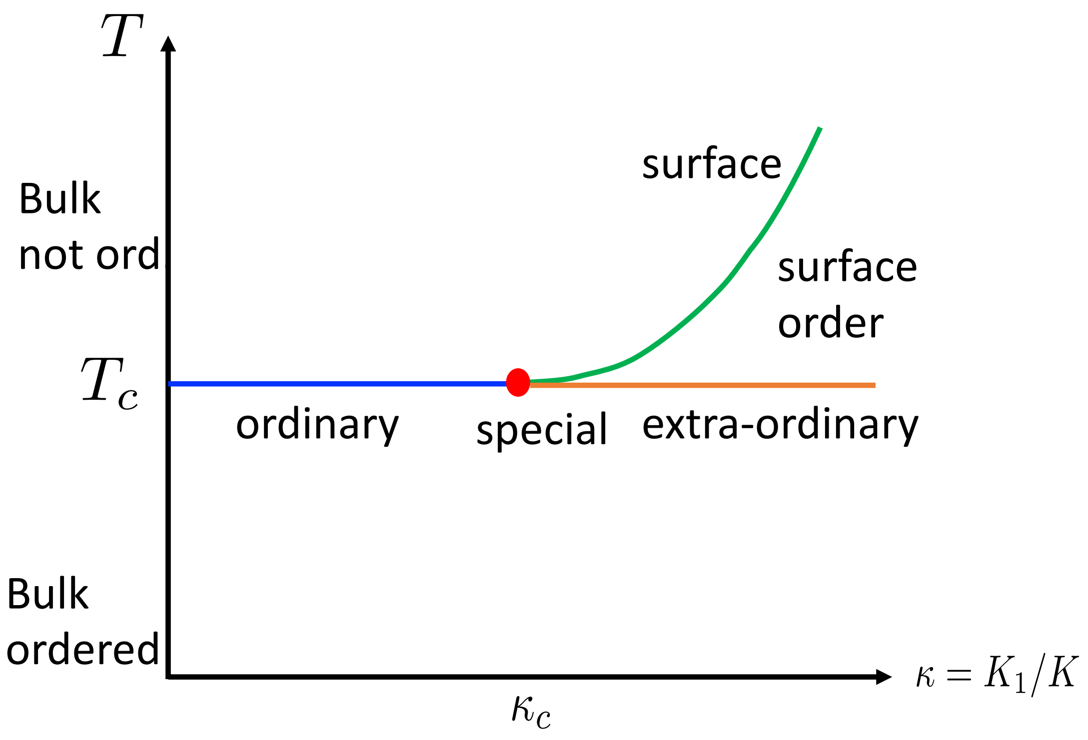
$d > 3$

“Normal” transition

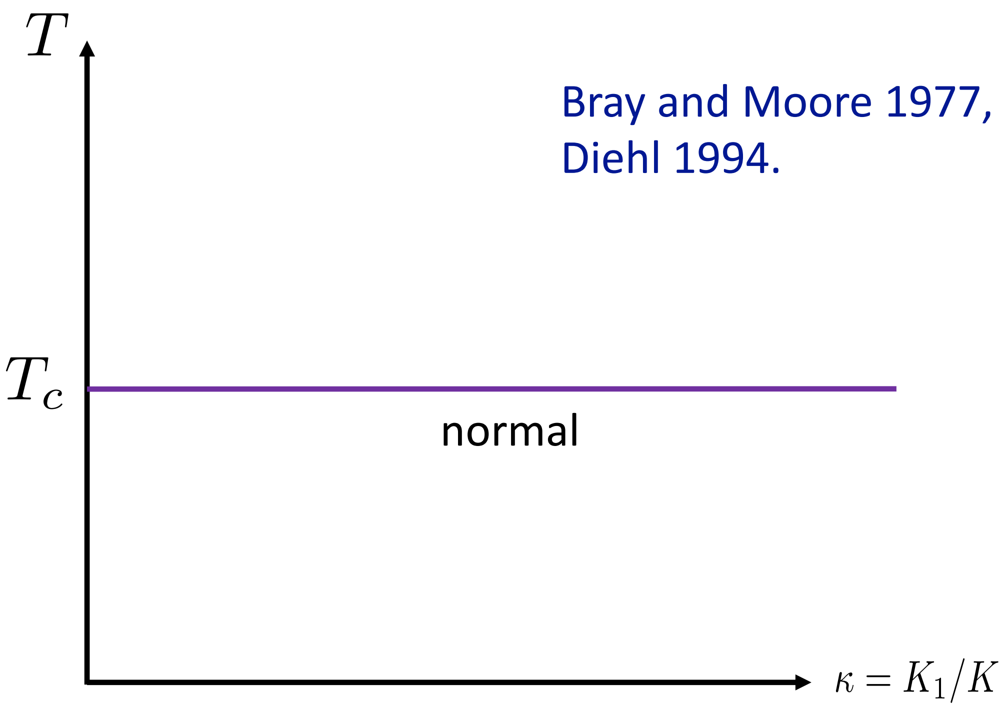
$$H = -\sum_{\langle ij \rangle} K_{ij} \vec{S}_i \cdot \vec{S}_j - \sum_{i \in surf} \vec{h}_1 \cdot \vec{S}_i$$

$d > 3$

Extra-ordinary = Normal



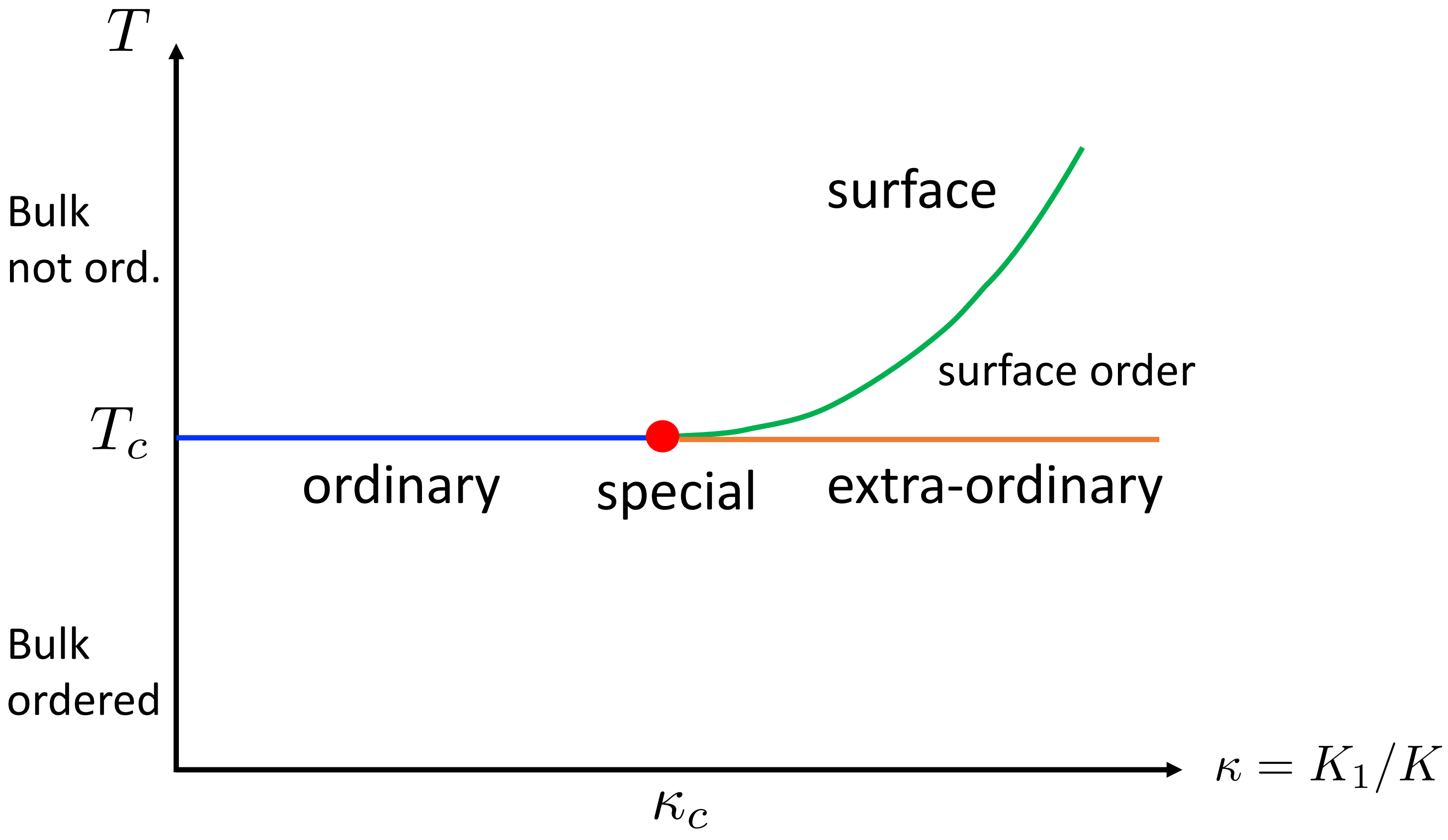
$h_1 = 0$



Bray and Moore 1977,
Diehl 1994.

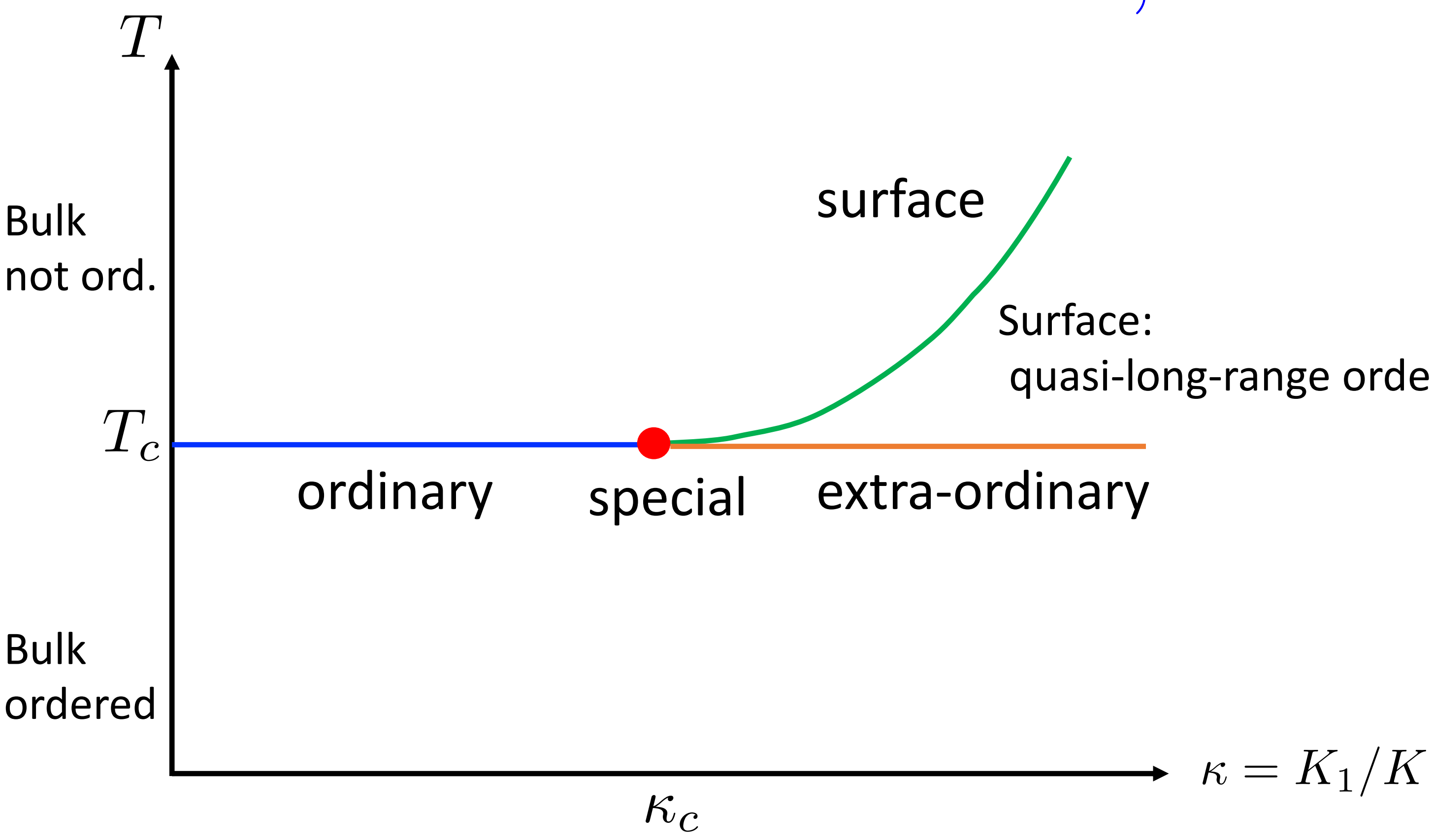
$h_1 \neq 0$

$$H = -\sum_{\langle ij \rangle} K_{ij} \vec{S}_i \cdot \vec{S}_j, \qquad d = 3$$



- $N = 1$

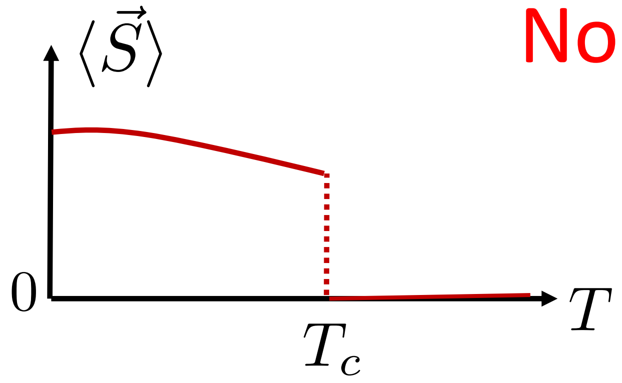
$d = 3, \;\; N = 2$



- Extra-ordinary??

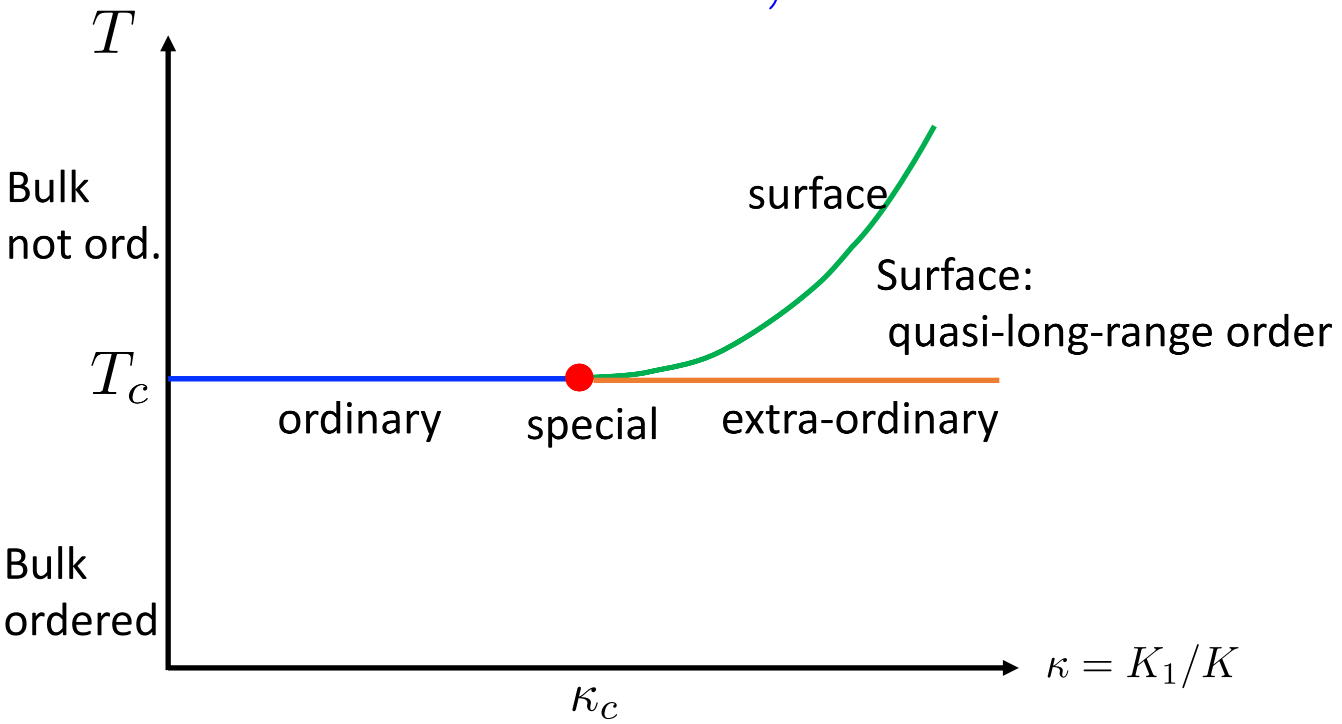
- Long range order at T_c ?

Deng, Blote, Nightingale, 2005



No!

$d = 3, \; N = 2$ extra-ordinary



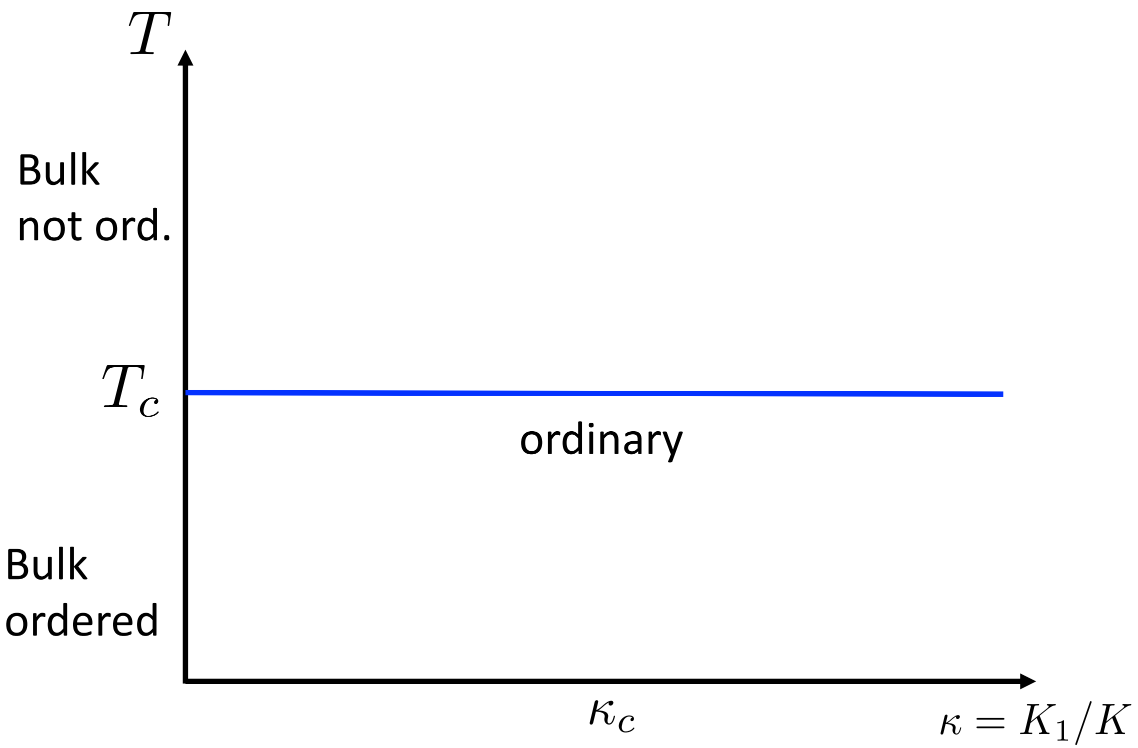
- $$\langle \vec{S}(x) \cdot \vec{S}(0) \rangle \sim \frac{1}{(\log x)^q}$$

"extra-ordinary-log"

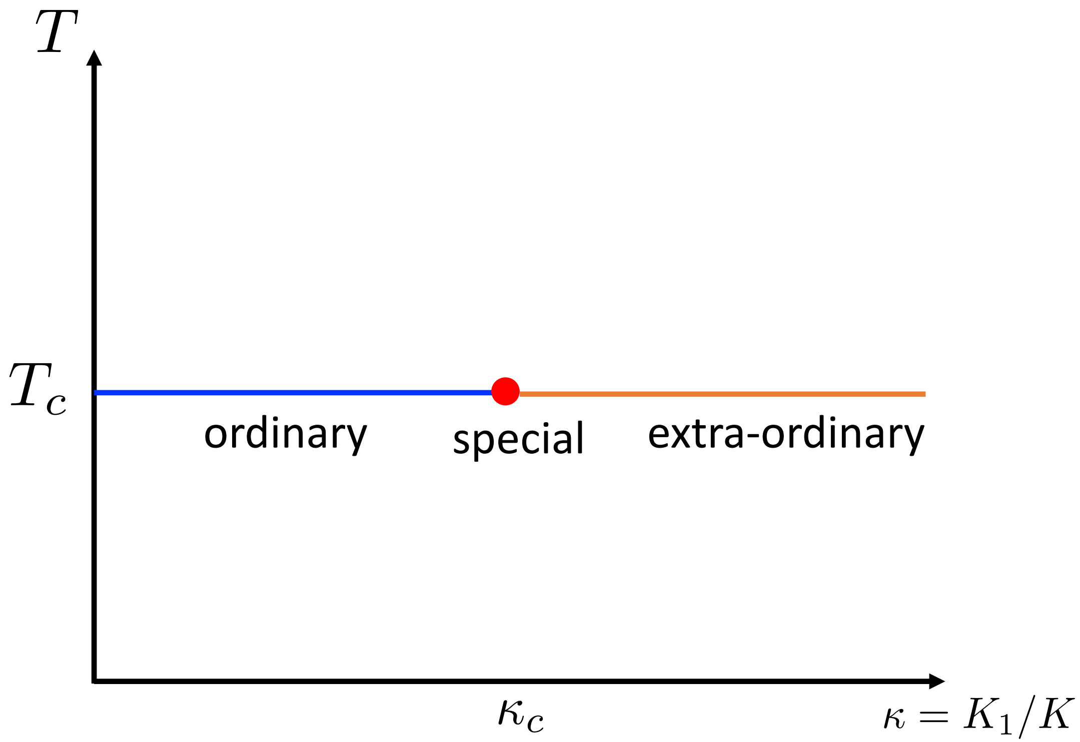
- $$\mathcal{K} \sim \frac{1}{2q} \log \xi_{bulk} \qquad \langle \vec{S}(\vec{x}) \cdot \vec{S}(0) \rangle \sim \frac{1}{|x|^{1/2\mathcal{K}}}$$

MM, 2020.

$$d = 3,\ N > 2$$

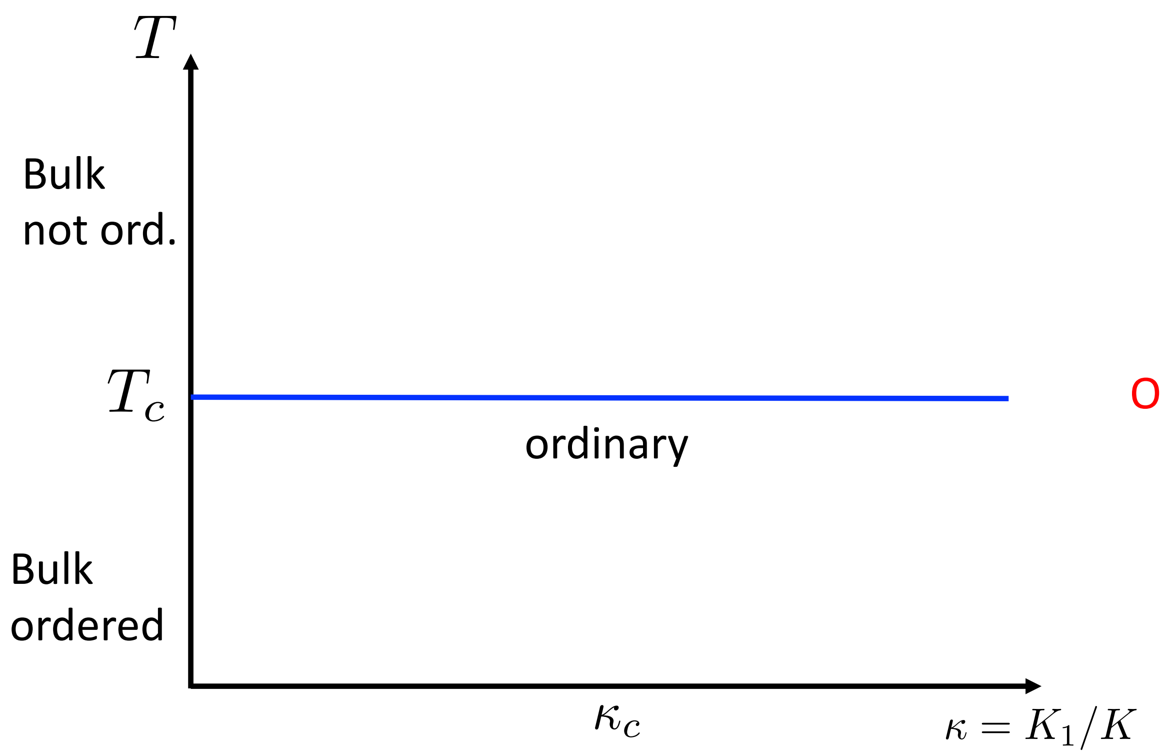

T
Bulk
not ord.
T_c
ordinary
Bulk
ordered
κ_c
$\kappa = K_1/K$

OR

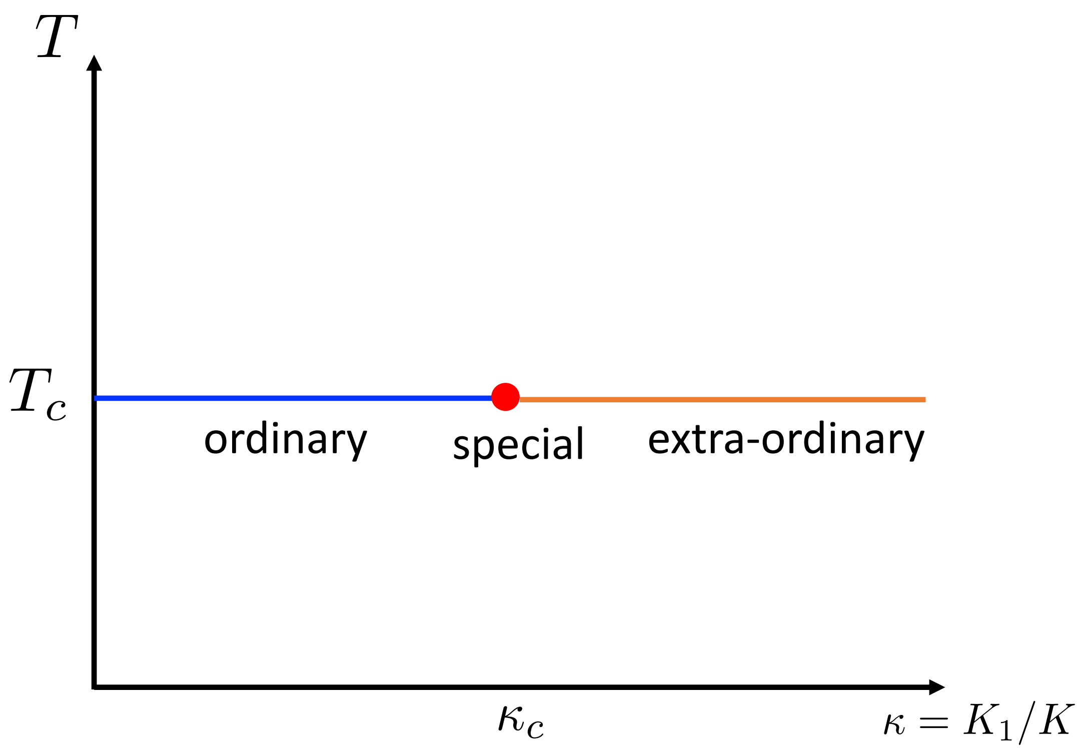

T
T_c
ordinary
special
extra-ordinary
κ_c
$\kappa = K_1/K$

?

$d = 3, \; N > 2$



OR



Large finite N

$$N \to 2^+$$

$$\langle \vec{S}(x) \cdot \vec{S}(0) \rangle \sim \frac{1}{(\log x)^q}$$

MM, 2020

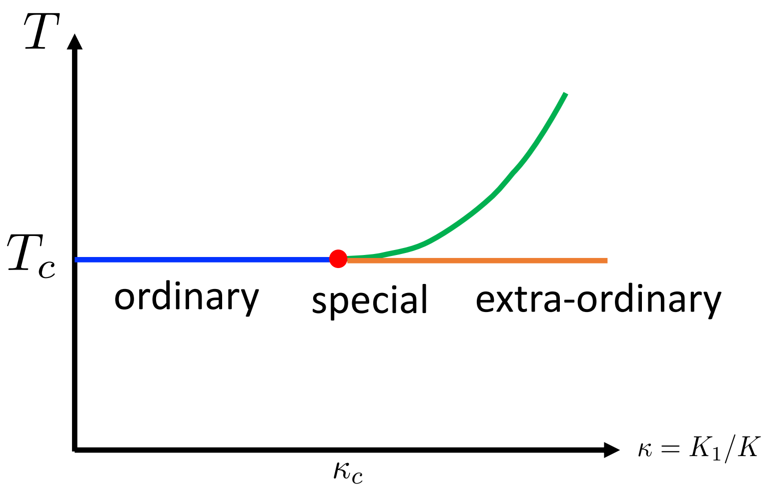
$d = 3,\ N \to \infty$

- No special fixed point at $N = \infty$ in $d = 3$

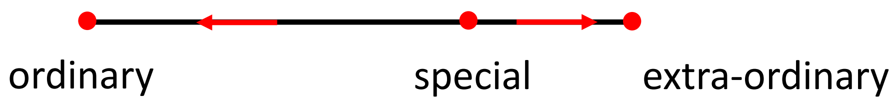
- $d = 3 + \epsilon$

$$(\Delta_{\vec{n}})_{spec} = \epsilon \left(1 + \frac{3}{N}\right) + O\left(\frac{1}{N^2}\right)$$

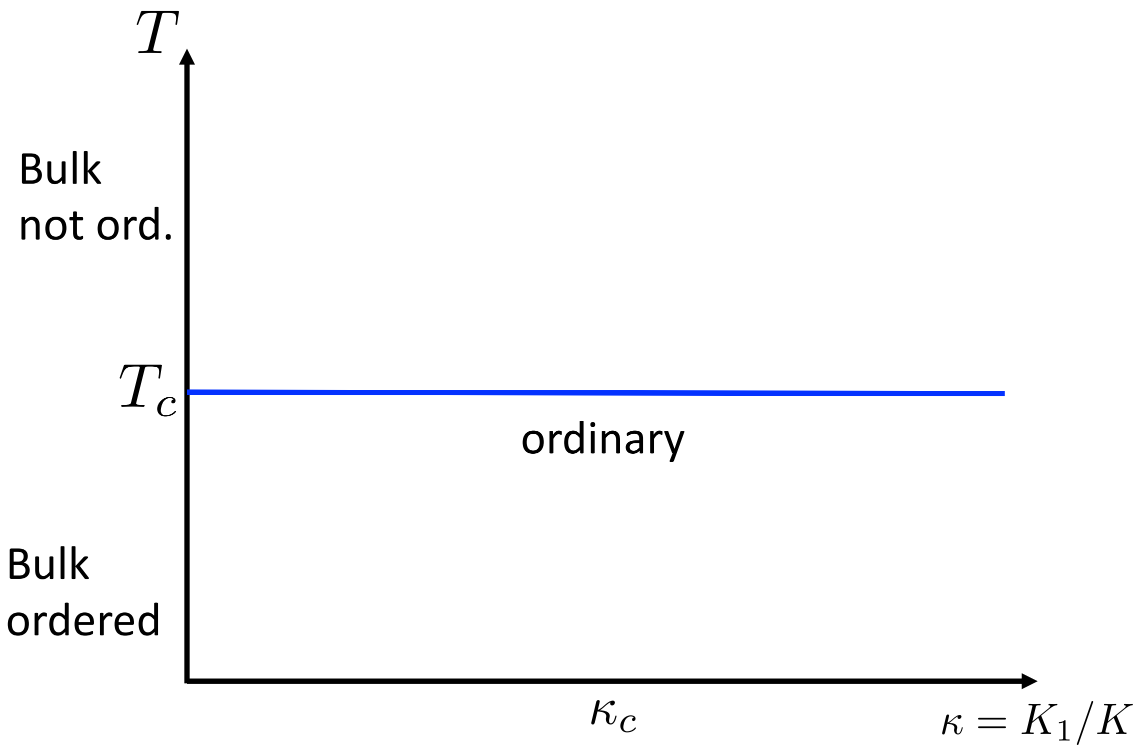
Ohno, Okabe 1983



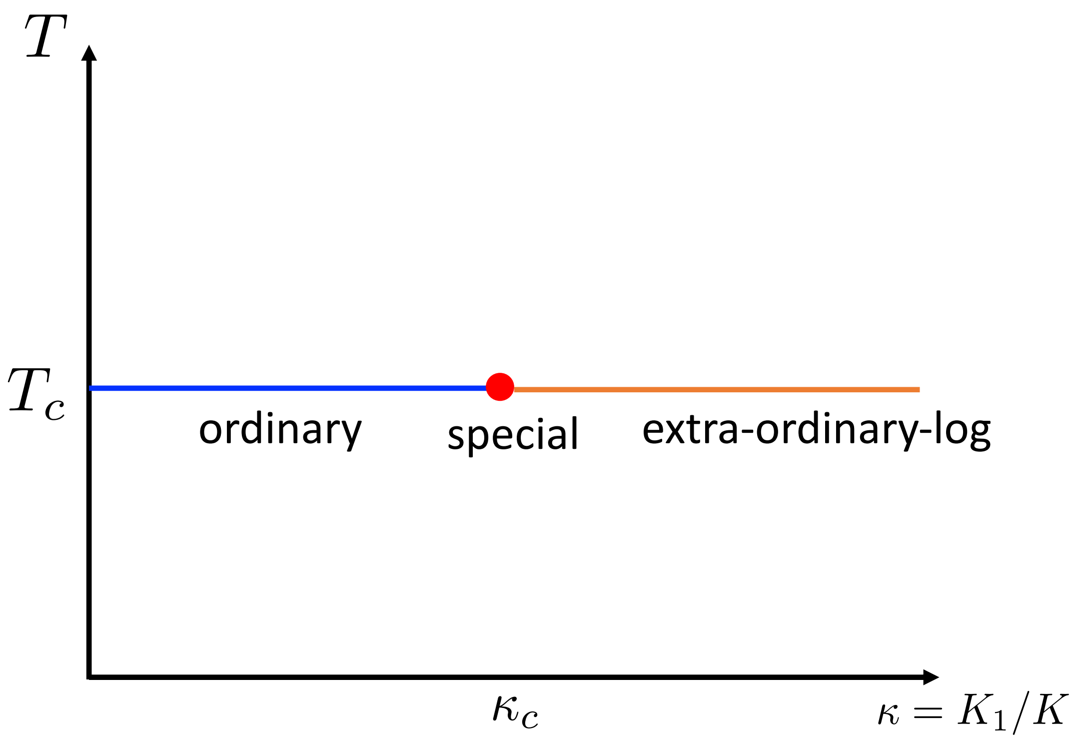
$d > 3$



$d = 3, \;\; N > 2$



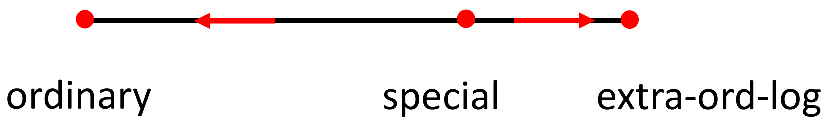
OR



Large finite N

$$N \to 2^+$$

$$\langle \vec{S}(x) \cdot \vec{S}(0) \rangle \sim \frac{1}{(\log x)^q}$$



RG sketch (d = 3)

- No bulk: $$S = \frac{1}{2g}\int d^2\mathrm{x}\ (\partial_\mu \vec{n})^2, \qquad \vec{n}^2 = 1$$

  Polyakov: $$\frac{dg}{d\ell} = \frac{N-2}{2\pi} g^2, \qquad \vec{n} \to \left(1 - \frac{\eta_n(g)}{2} d\ell\right)\vec{n}, \qquad \eta_n = \frac{N-1}{2\pi} g$$

- Coupling to bulk: $$\frac{dg}{d\ell} = -\alpha g^2, \qquad \alpha = \frac{\pi s^2}{2} - \frac{N-2}{2\pi}$$

$$\alpha > 0, \qquad g \to 0$$

$$\langle \vec{n}(x)\cdot\vec{n}(0)\rangle \sim \frac{1}{(\log x)^q}$$

- Extra-ordinary-log fixed point

$$q = \frac{N-1}{2\pi\alpha}$$

RG sketch (d = 3)

- No bulk: $$S = \frac{1}{2g}\int d^2\mathrm{x}\ (\partial_\mu \vec{n})^2, \qquad \vec{n}^2 = 1$$

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- Coupling to bulk: $$\frac{dg}{d\ell} = -\alpha g^2, \qquad \alpha = \frac{\pi s^2}{2} - \frac{N-2}{2\pi}$$

$\alpha > 0, \qquad g \to 0$ - Extra-ordinary-log fixed point

$\alpha < 0, \qquad g = 0$ - unstable

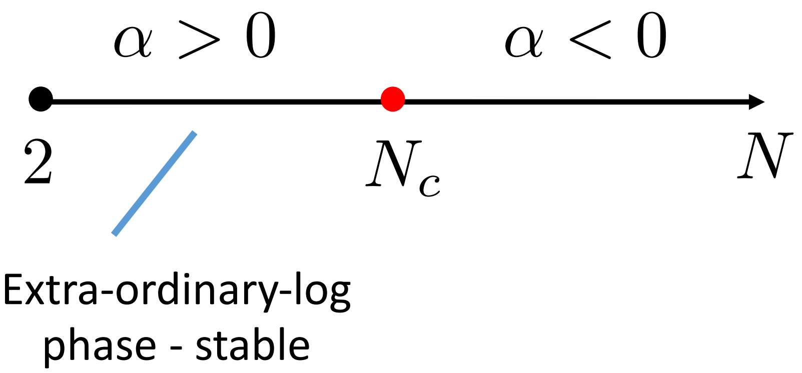
N_c

$$\frac{dg}{d\ell} = -\alpha g^2,$$

$$\alpha = \frac{\pi s^2}{2} - \frac{N-2}{2\pi}$$

$$\alpha(N=2) = \frac{\pi s^2}{2} > 0$$

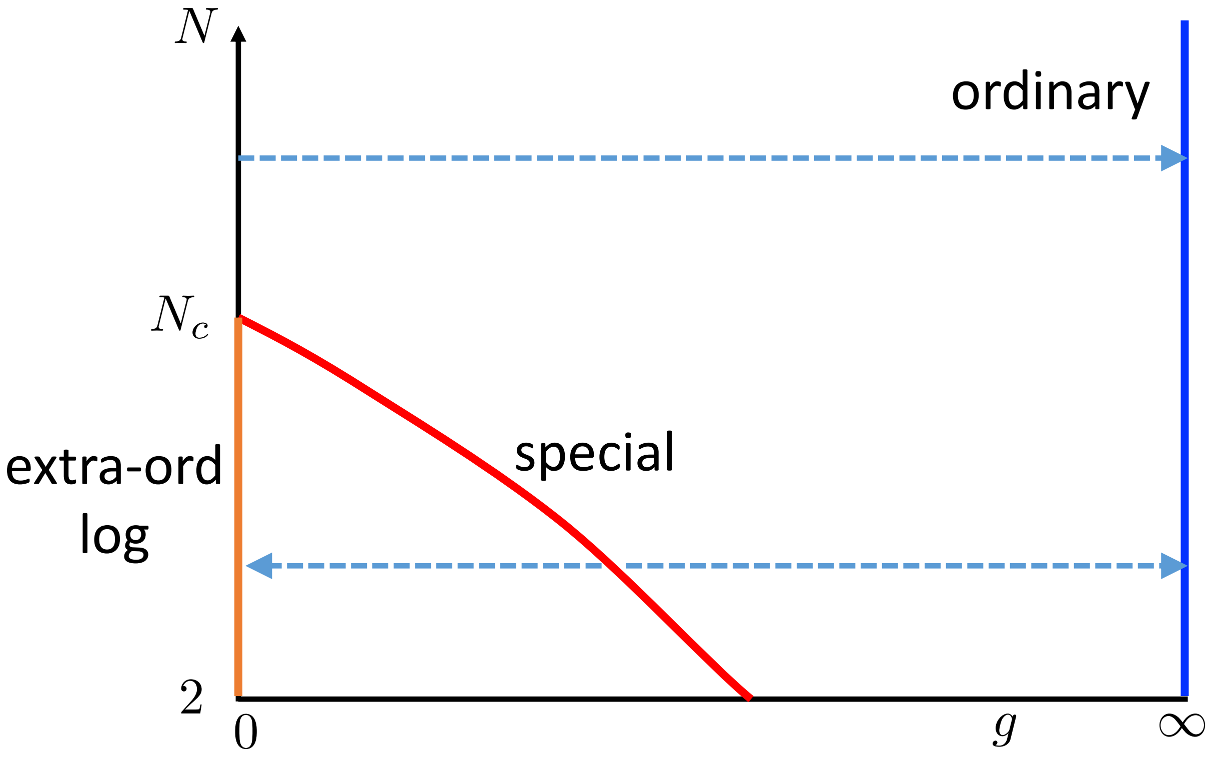
$$\alpha(N \to \infty) \approx -\frac{N-4}{4\pi} < 0$$



- Large-N: $N_c \approx 4$

Near N_c

$$\frac{dg}{d\ell} \approx a(N - N_c)g^2 + bg^3, \qquad a > 0$$



Scenario I: $b > 0$

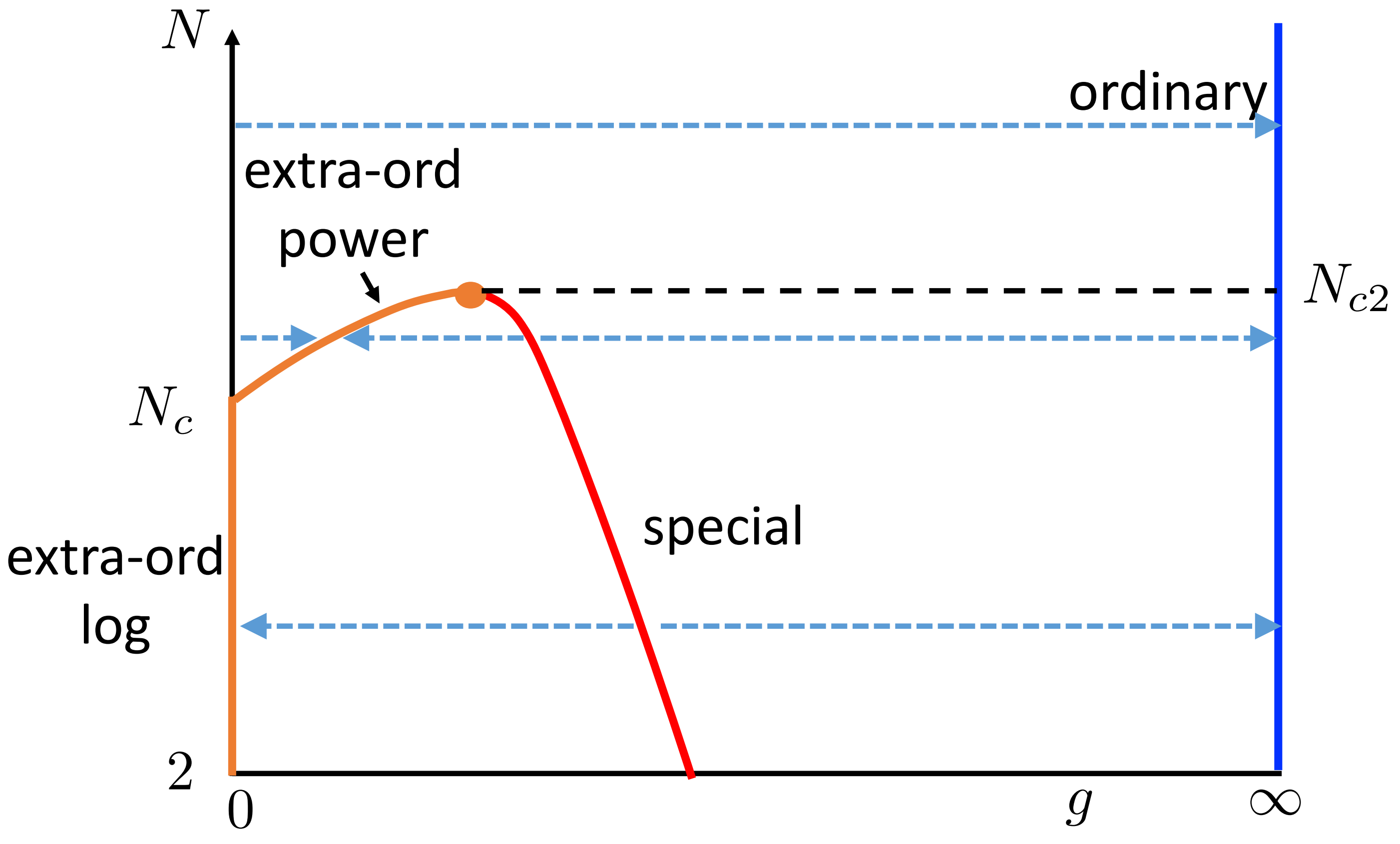
$$N \to N_c^-, \qquad g_*^{spec} \sim \frac{a(N_c - N)}{b}$$

$$\Delta_{\vec{n}} \approx \frac{N-1}{4\pi} g_*$$

$$\nu^{-1} = \frac{a^2(N_c - N)^2}{b}$$

Near N_c

$$\frac{dg}{d\ell} \approx a(N - N_c)g^2 + bg^3, \qquad a > 0$$



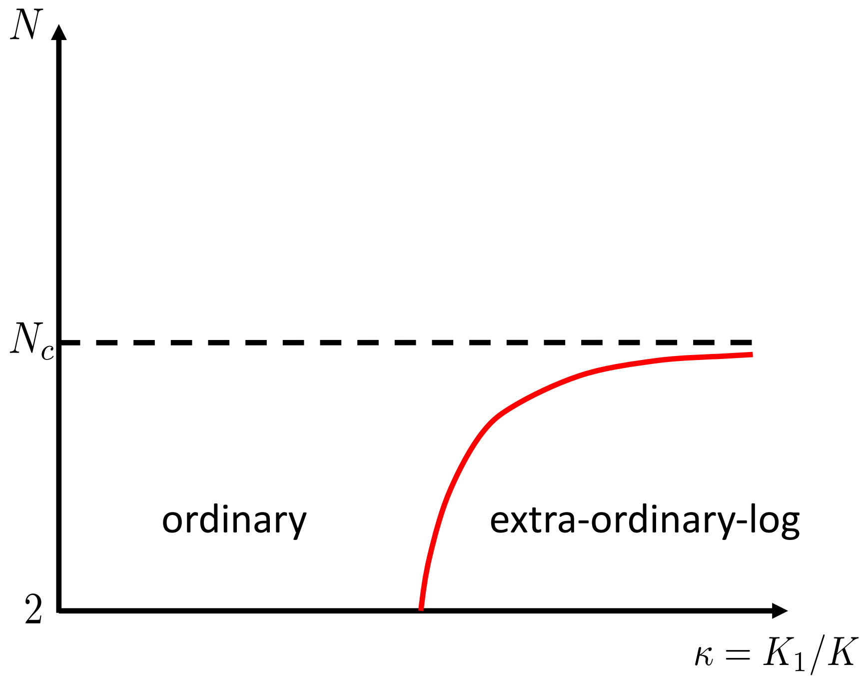
Scenario II: $b < 0$

$$N \to N_c^+, \qquad g_* = \frac{a(N - N_c)}{|b|}$$

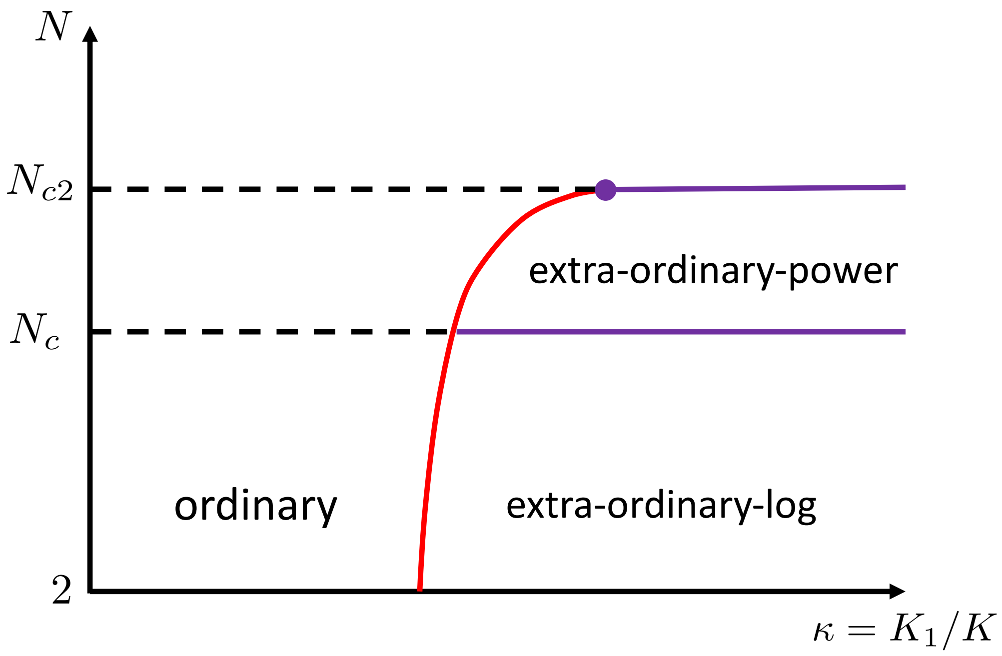
$$\Delta_{\vec{n}} \approx \frac{N-1}{4\pi} g_*$$

“Extra-ordinary-power” class.

$$d = 3,\ \ N > 2$$



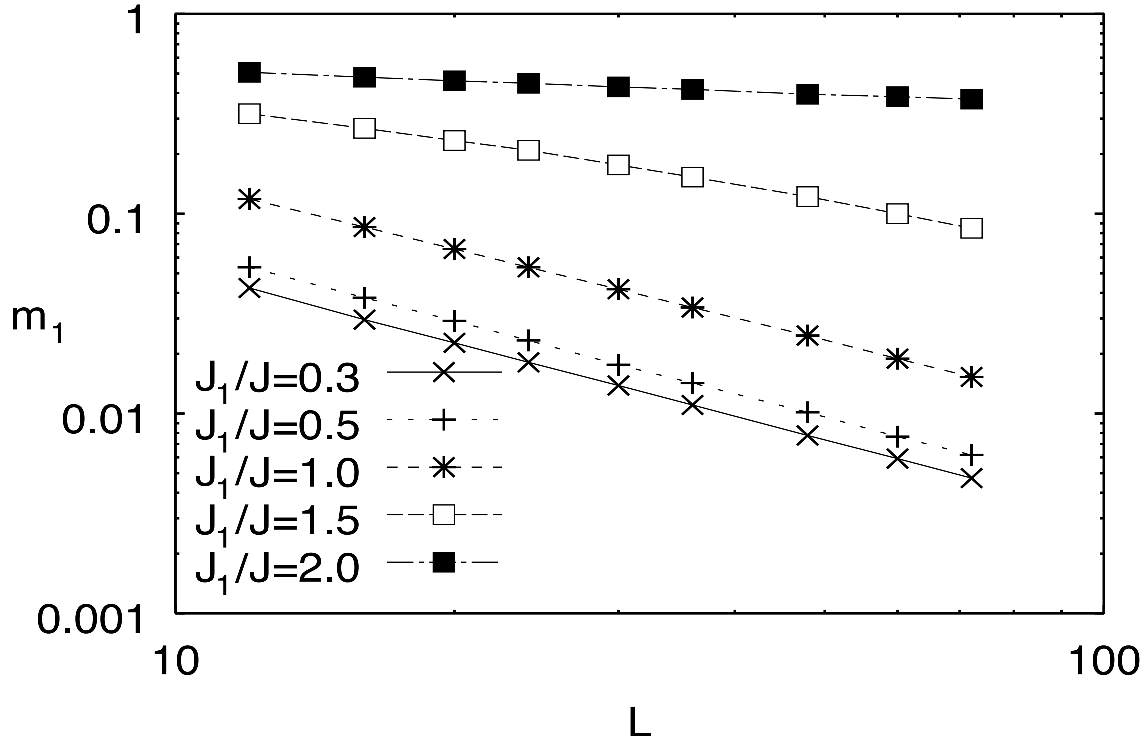
Scenario I

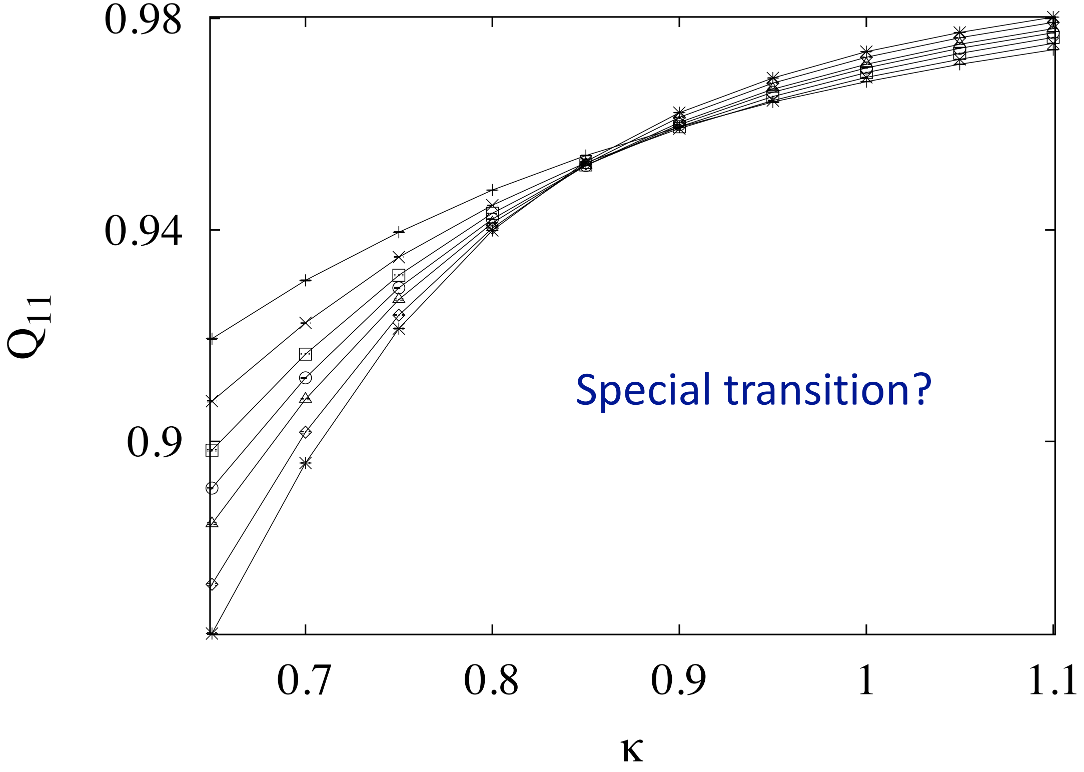


Scenario II

Numerics: N = 3

- Classical Monte-Carlo



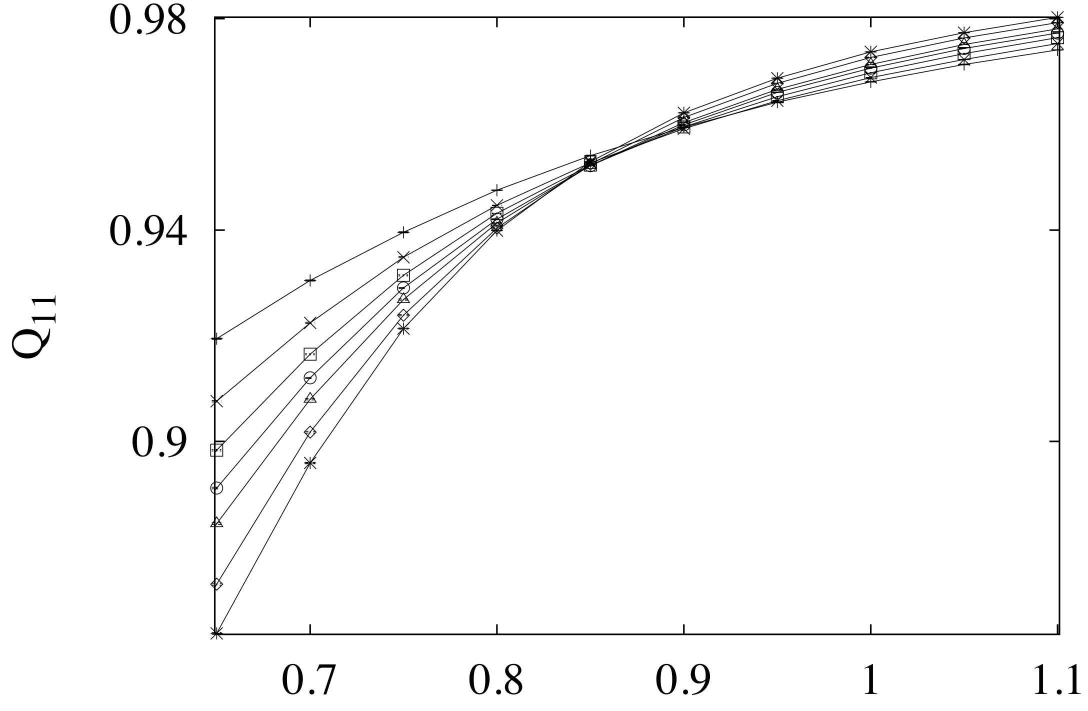


$$Q_{11} = \frac{\langle \vec{m}_1 \cdot \vec{m}_1 \rangle^2}{\langle (\vec{m}_1 \cdot \vec{m}_1)^2 \rangle}$$

M. Krech, 2000

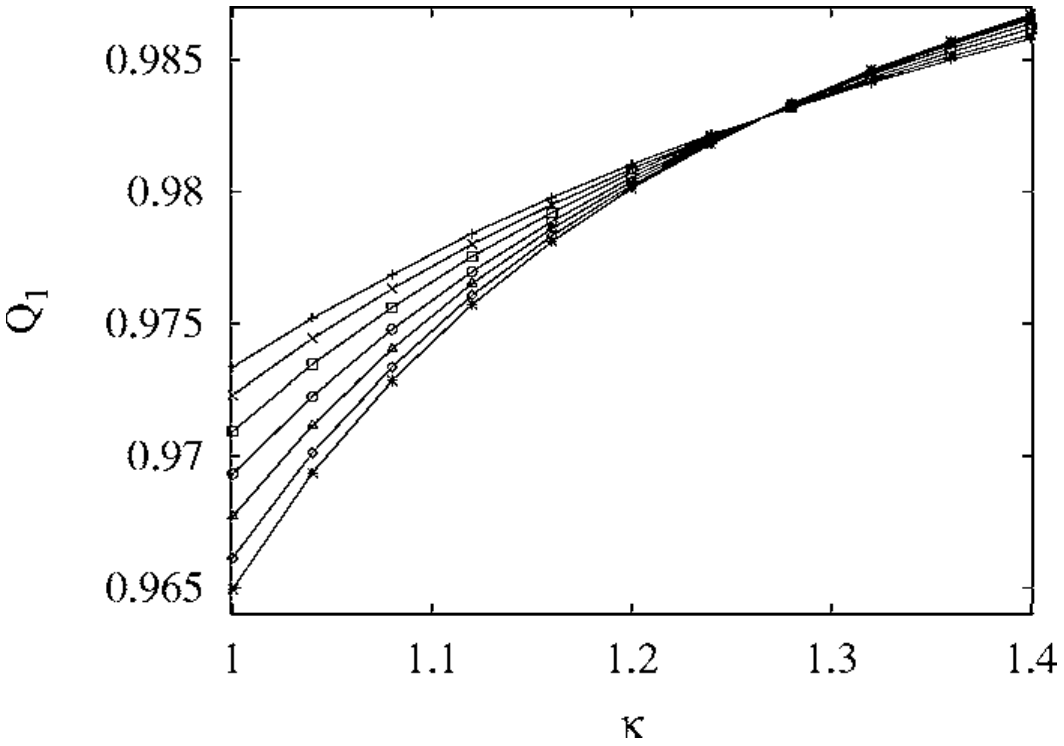
Deng, Blote, Nightingale, 2005

N = 3



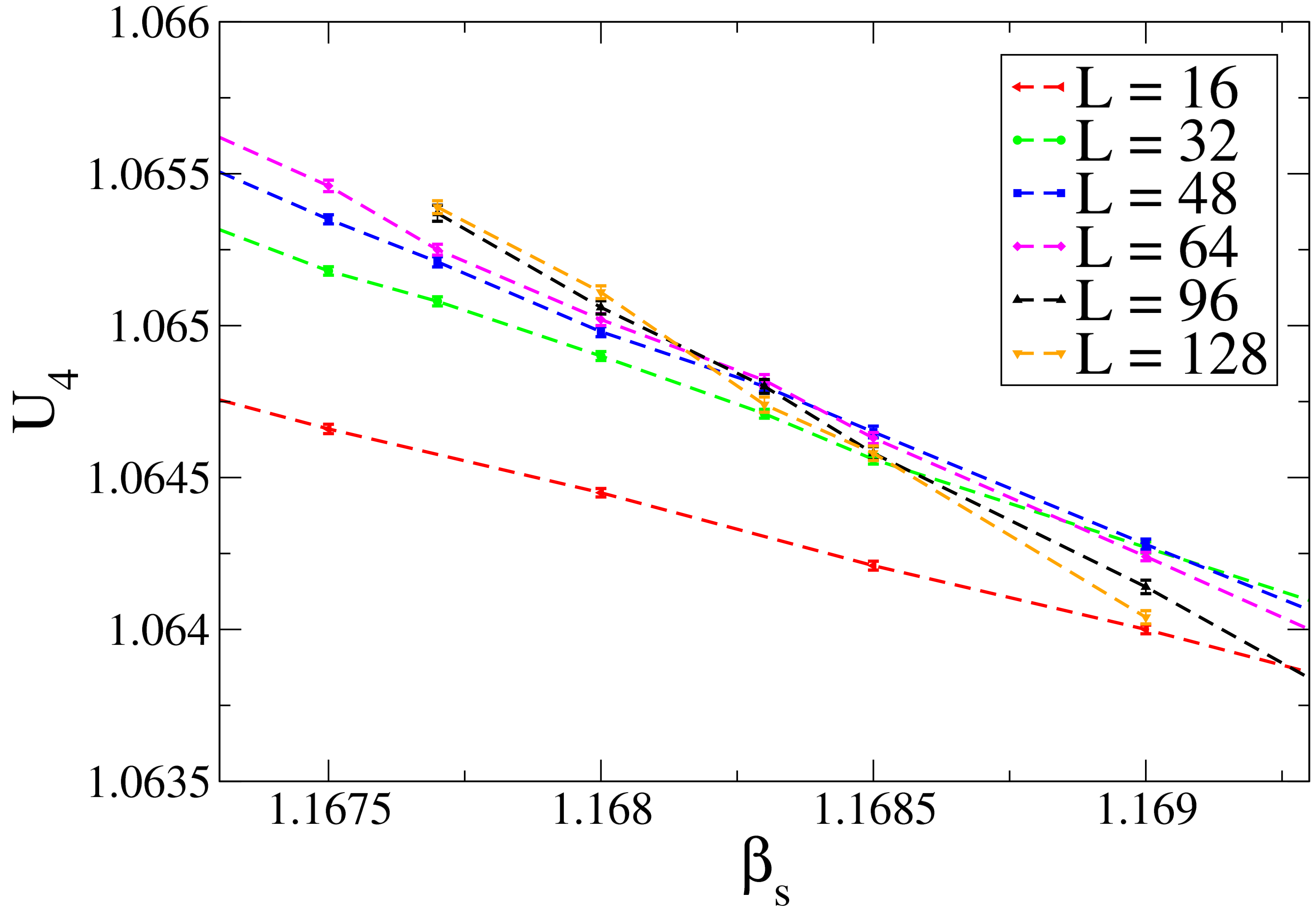
Deng, Blote, Nightingale, 2005

N = 4



Deng, 2006

Classical model: N = 3

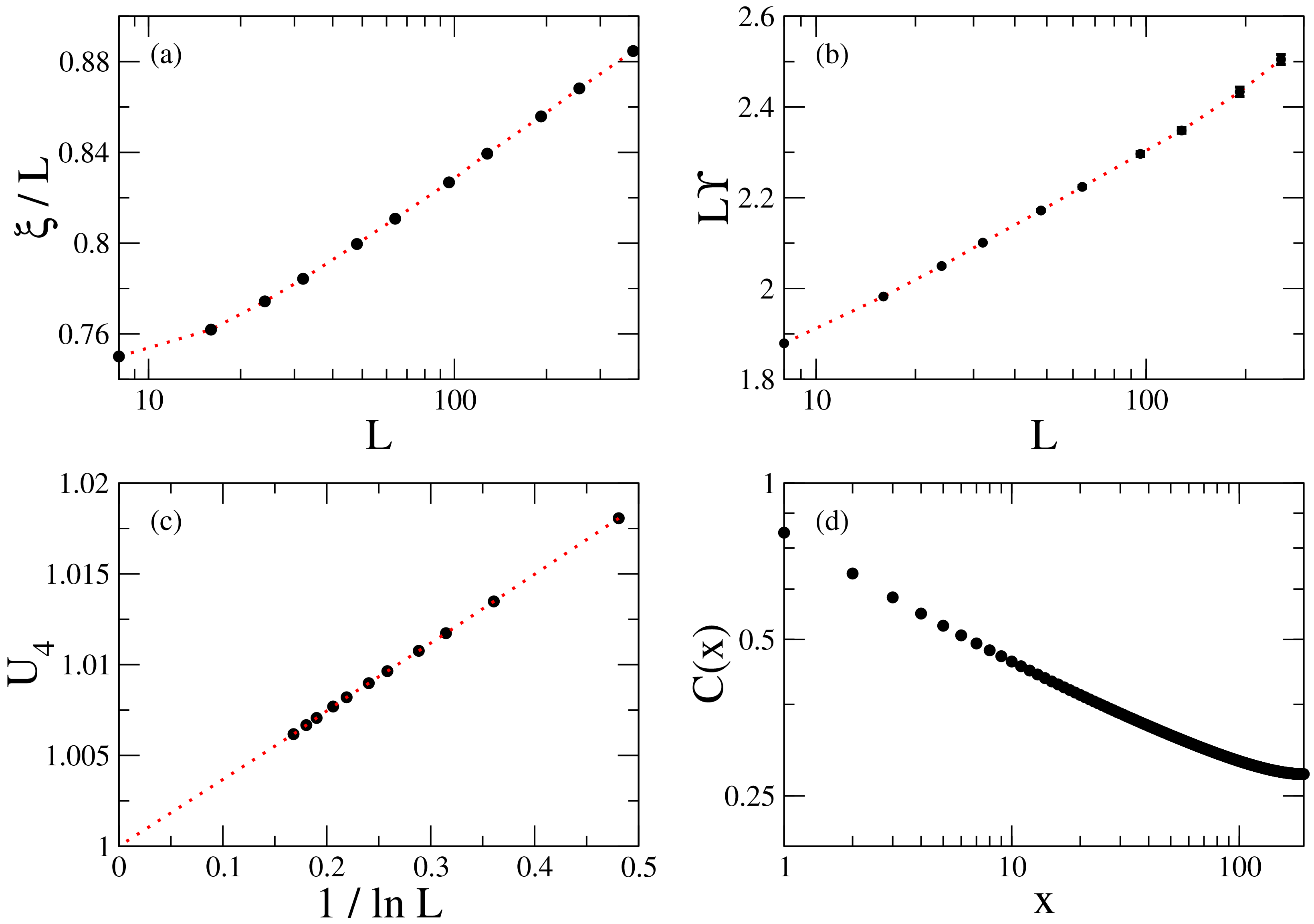


$$\nu_{spec}^{-1} \approx 0.36$$

$$(\Delta_{\vec{n}})_{spec} \approx 0.26$$

F. P. Toldin, 2020

Classical model: N = 3 – extra-ordinary log phase?



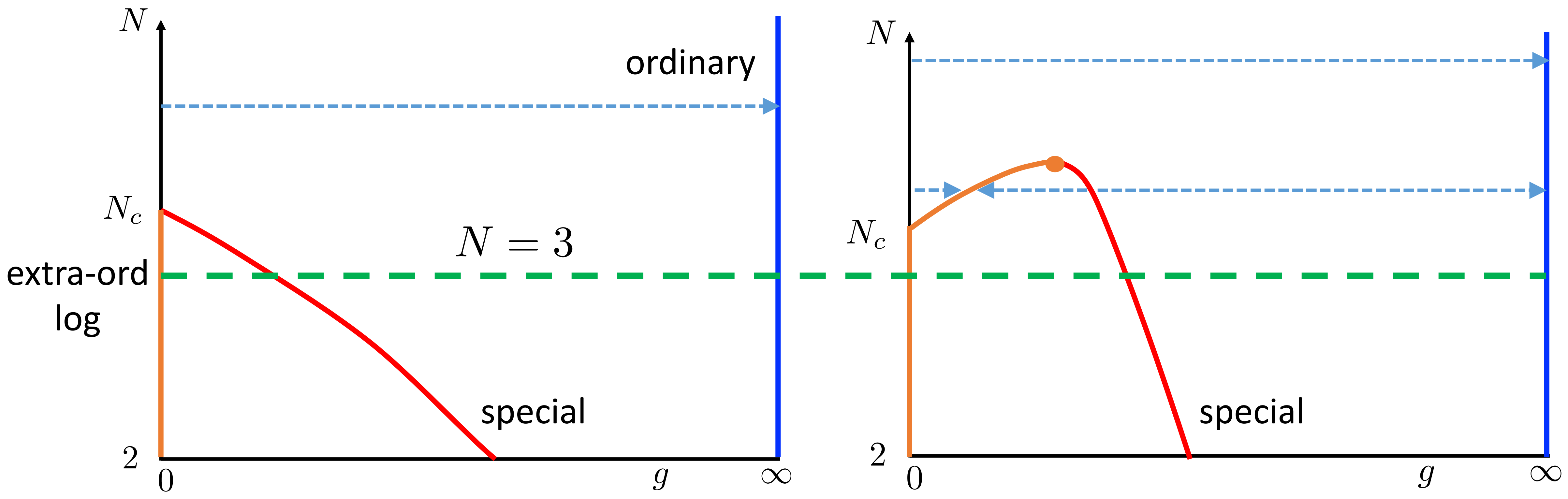
$$L\Upsilon \approx \frac{1}{g(L)} \approx \alpha \log L$$

$$C(x) = \langle \vec{S}(x) \cdot \vec{S}(0) \rangle \sim \frac{1}{(\log x)^q}$$

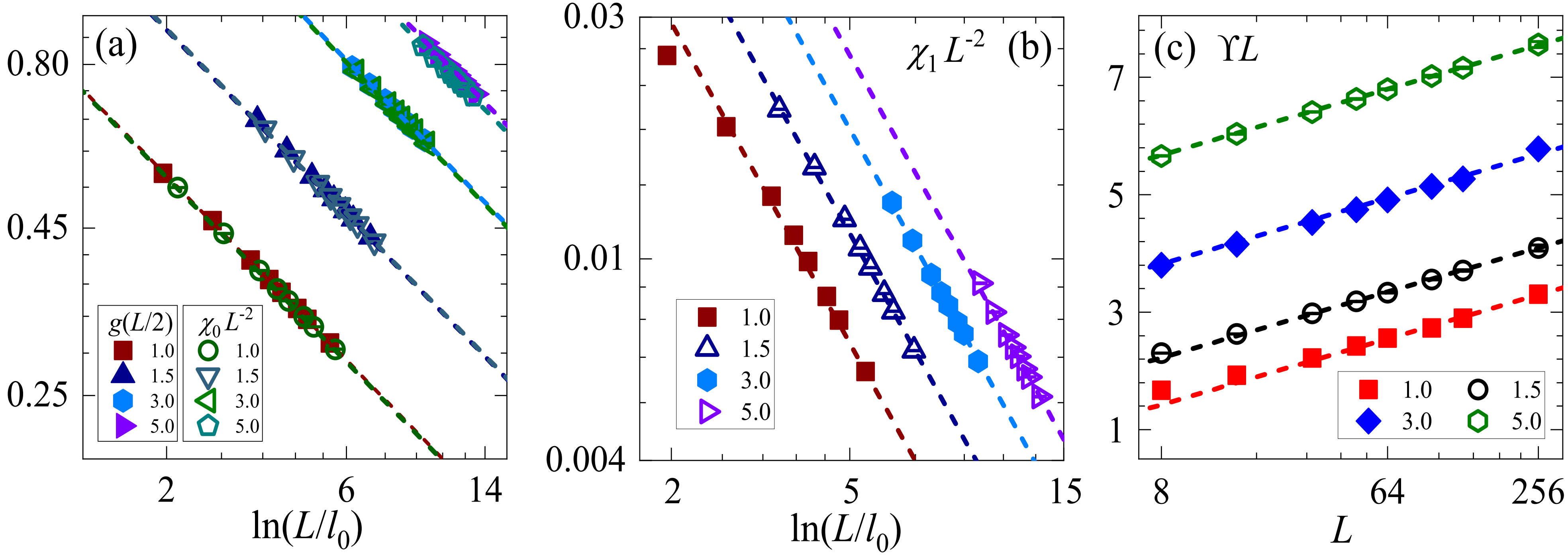
$$\alpha \approx 0.11 - 0.15$$

F. P. Toldin, 2020

Classical phase diagram



Classical model: N = 2 – extra-ordinary log phase?



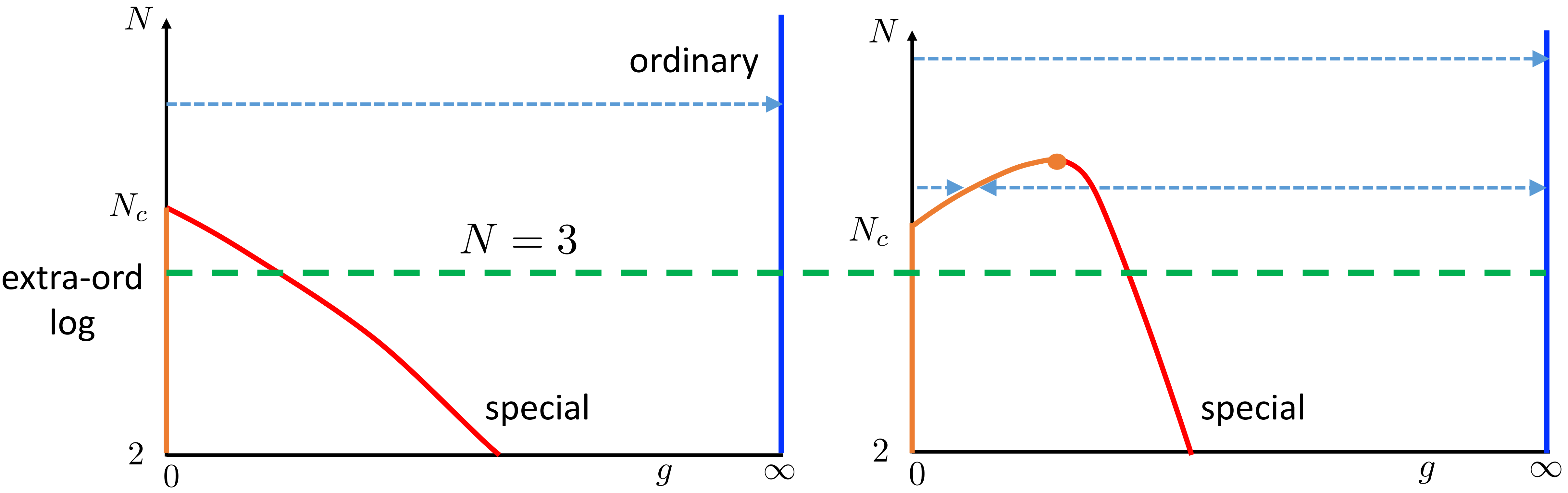
$$C(x) = \langle \vec{S}(x) \cdot \vec{S}(0) \rangle$$
$$\sim \frac{1}{(\log x)^q}$$

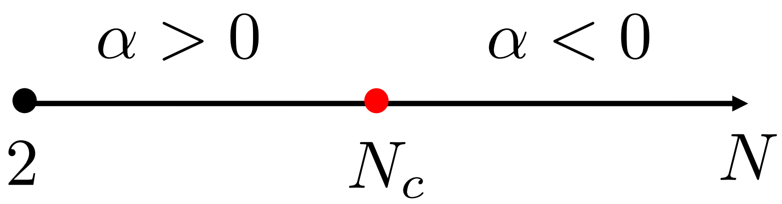
$$L\Upsilon \approx \frac{1}{g(L)} \approx \alpha \log L$$

$$q = \frac{N-1}{2\pi\alpha}, \quad \alpha \approx 0.27$$

M. Hu, Y. Deng and J.-P. Lv, 2020

Classical phase diagram



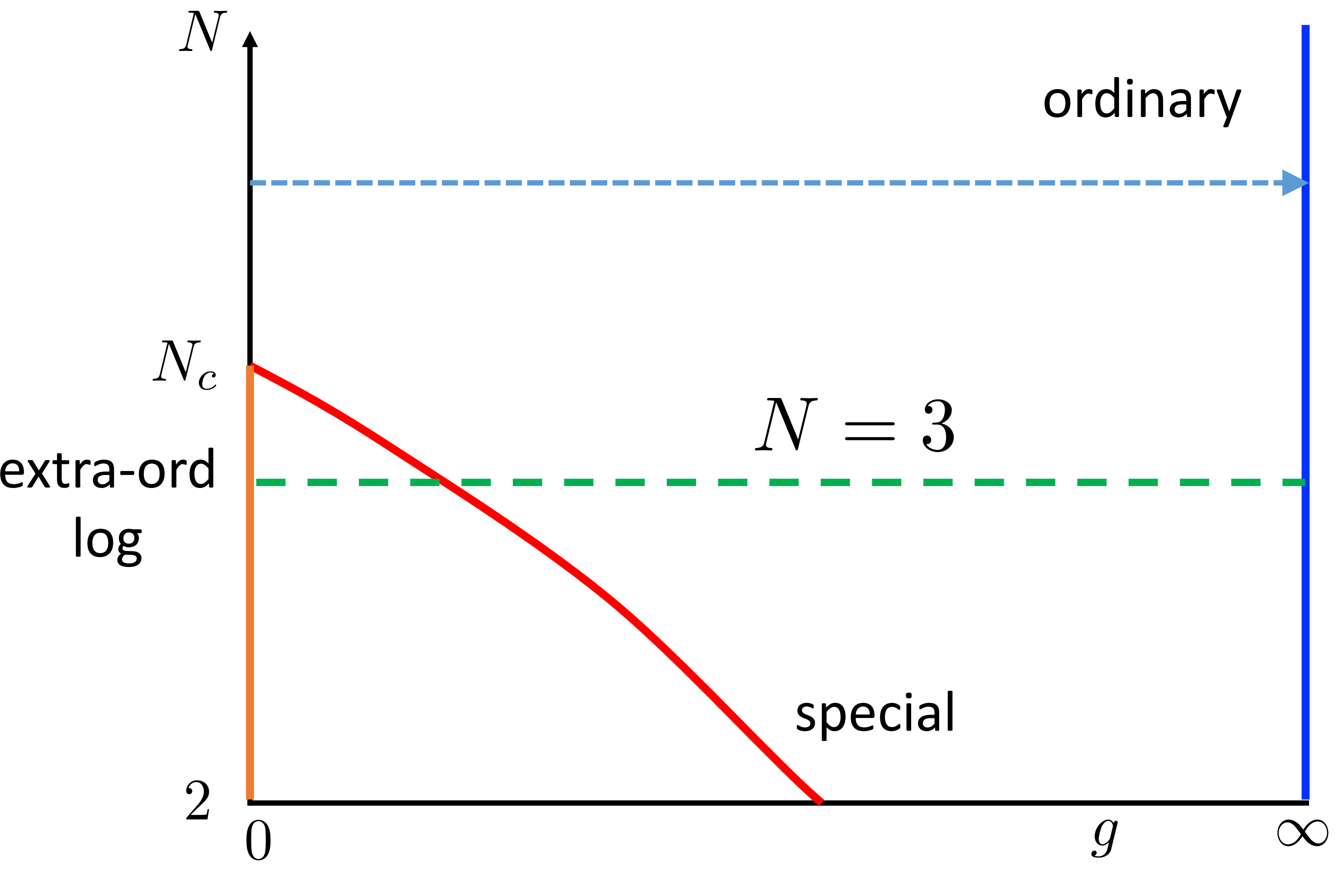


N	α
2	0.27
3	0.11-0.15

$N \to N_c$

$$\frac{dg}{d\ell} \approx a(N - N_c)g^2 + bg^3, \qquad a > 0$$

Scenario I: $b > 0$



$$N \to N_c^-, \quad g_*^{sp} = \frac{a(N_c - N)}{b}$$

$$(\Delta_{\vec{n}})_{spec} \approx \frac{N-1}{4\pi} g_*^{sp}$$

$$\nu_{spec}^{-1} \approx \frac{a^2(N_c - N)^2}{b}$$

Special transition (classical MC)

N	ν^{-1}	$\Delta_{\vec{n}}$
1	0.72	0.36
2	0.61	0.33
3	0.36	0.26
4	0.1	0.14

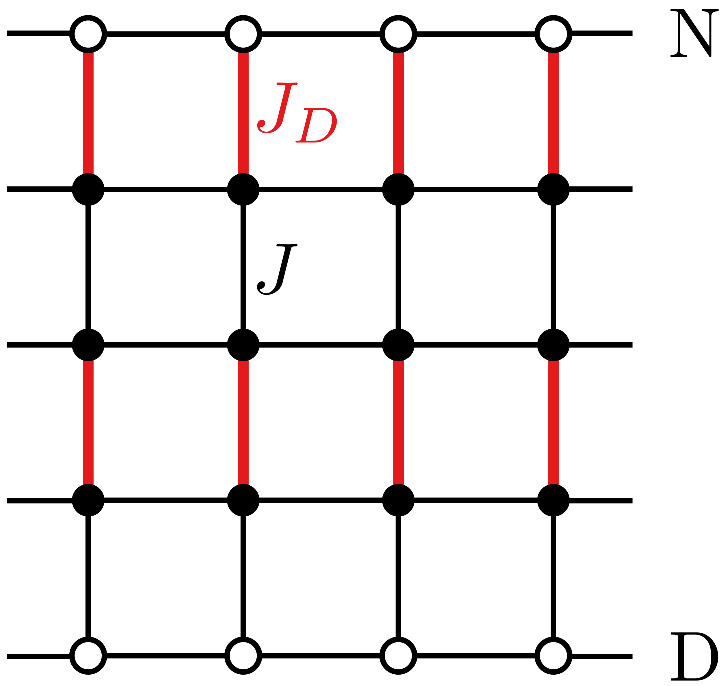
$$N \to N_c^-, \quad g_*^{sp} = \frac{a(N_c - N)}{b}$$

$$(\Delta_{\vec{n}})_{spec} \approx \frac{N-1}{4\pi} g_*^{sp}$$

$$\nu_{spec}^{-1} \approx \frac{a^2(N_c - N)^2}{b}$$

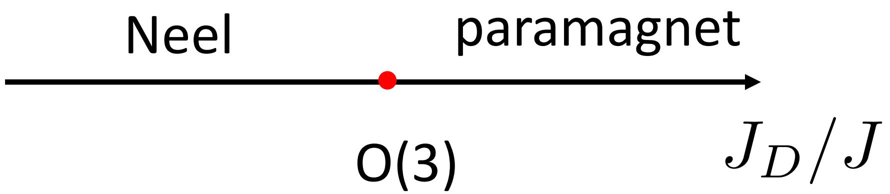
Deng, Blote, Nightingale, 2005
Deng, 2006
Toldin, 2020

2+1D quantum spin models, N = 3



$$H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

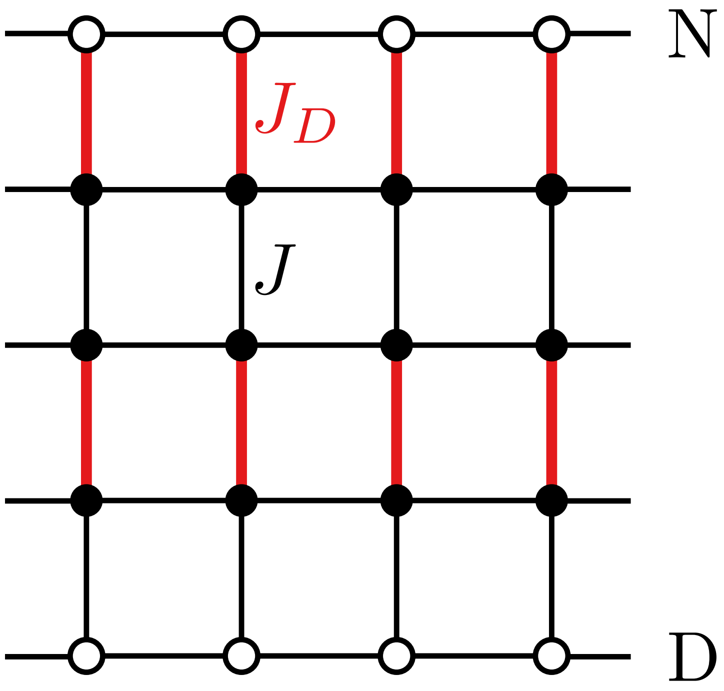
Bulk:



M. Matsumoto, C. Yasude, S. Todo et al, 2001

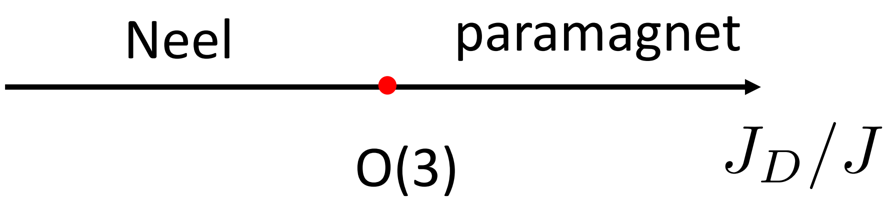
L. Zhang and F. Wang, 2017
C. Ding, L. Zhang and W. Guo, 2018
L. Weber, F. Parisen Toldin, S. Wessel, 2018
L. Weber and S. Wessel, 2019, 2020
W. Zhu, C. Ding, L. Zhang and W. Guo, 2020

2+1D quantum spin models, N = 3



$$H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

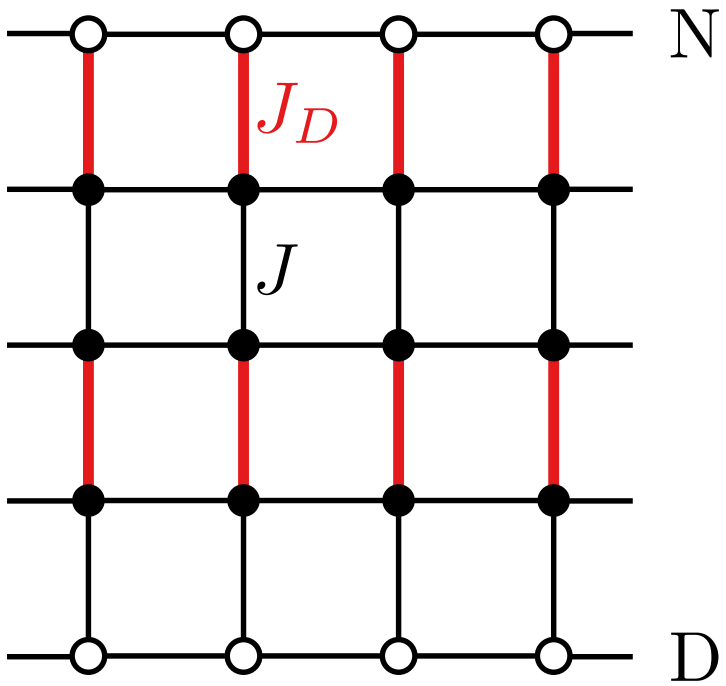
Bulk:

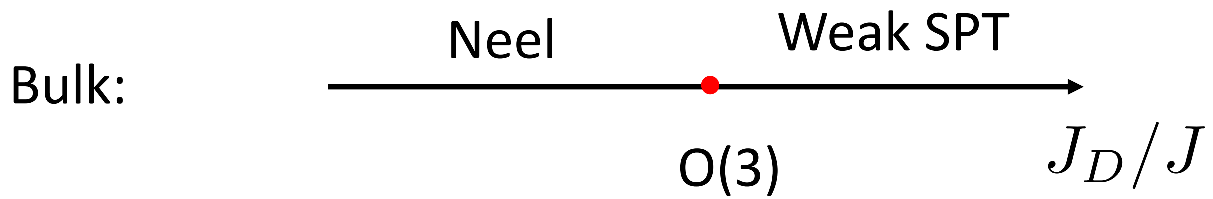


Non-dangling edge: $\Delta_{\vec{n}} \approx 1.15$ (Ordinary)

Dangling edge: $\Delta_{\vec{n}} \approx 0.25$

S = 1/2: weak SPT

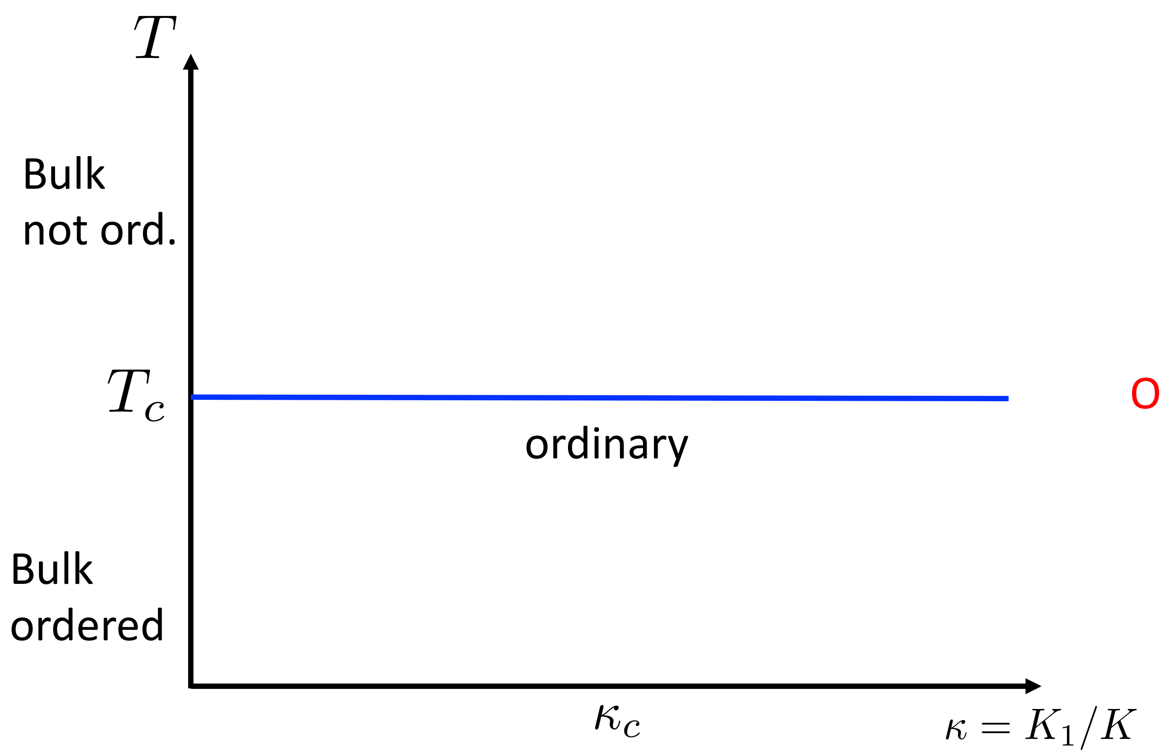




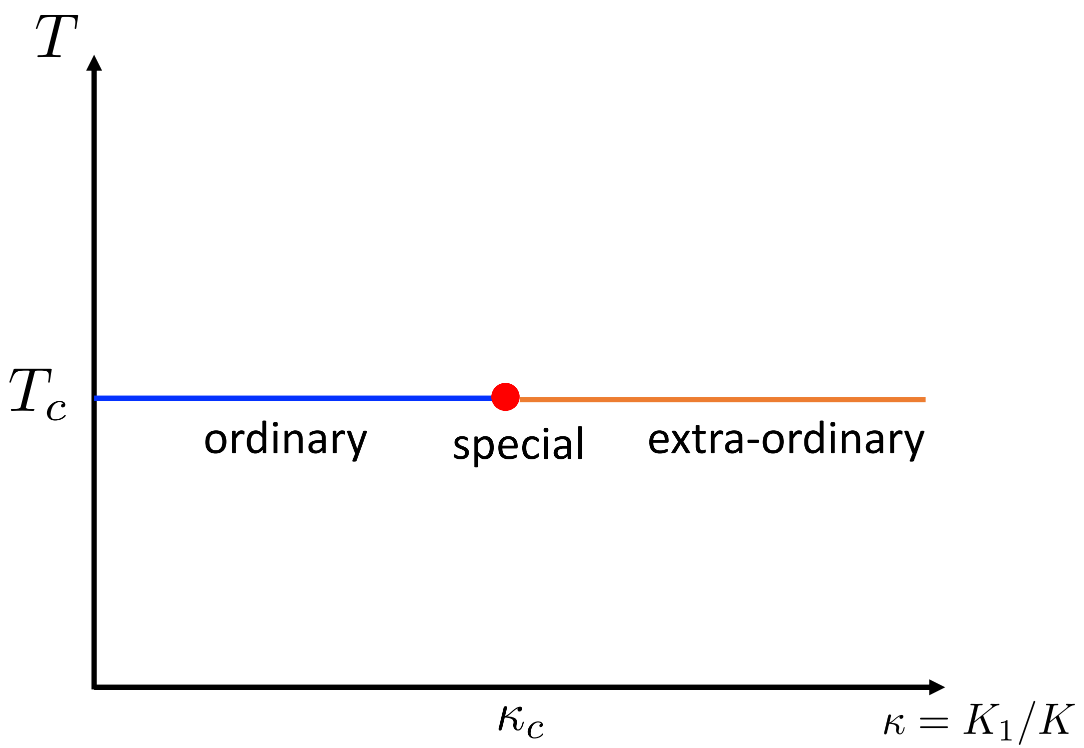
Non-dangling edge: $\Delta_{\vec{n}} \approx 1.15$ (Ordinary)

Dangling edge: $\Delta_{\vec{n}} \approx 0.25$

$d = 3, \ N > 2$



OR



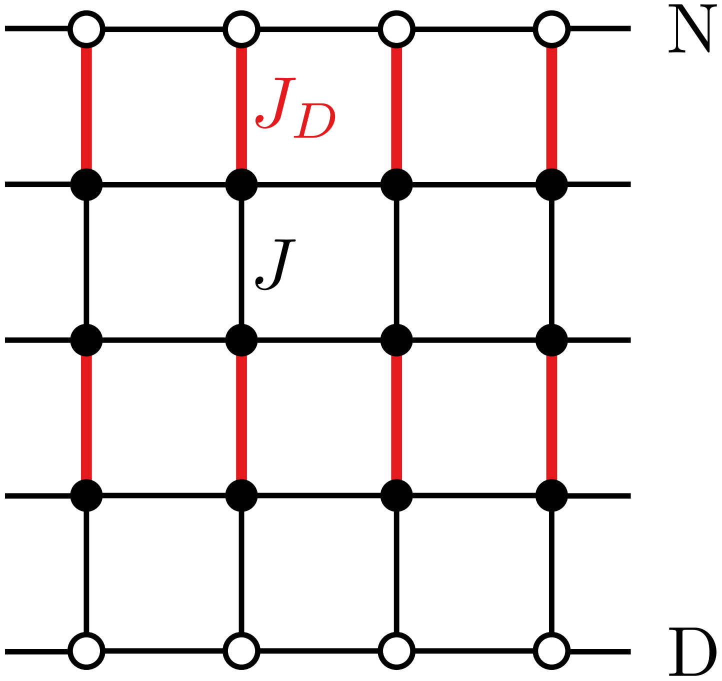
Large finite N

$N \to 2^+$

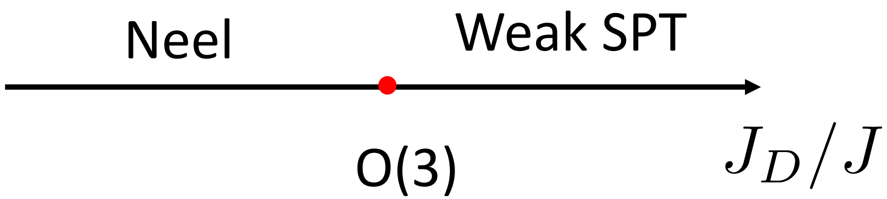
$$\langle \vec{S}(x) \cdot \vec{S}(0) \rangle \sim \frac{1}{(\log x)^q}$$

MM, 2020

S = 1/2: weak SPT



Bulk:



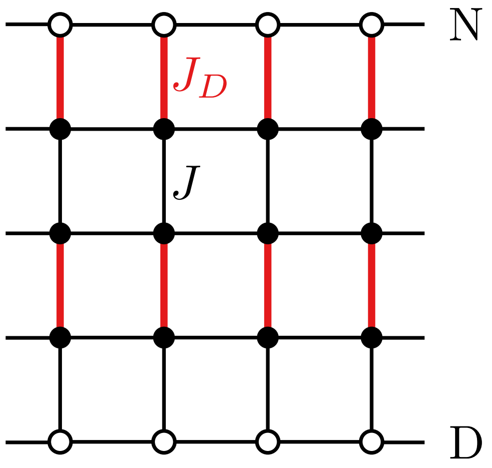
Non-dangling edge: $\Delta_{\vec{n}} \approx 1.15$ (Ordinary)

Dangling edge: $\Delta_{\vec{n}} \approx 0.25$

Similar exponents for S = ½ and S = 1!

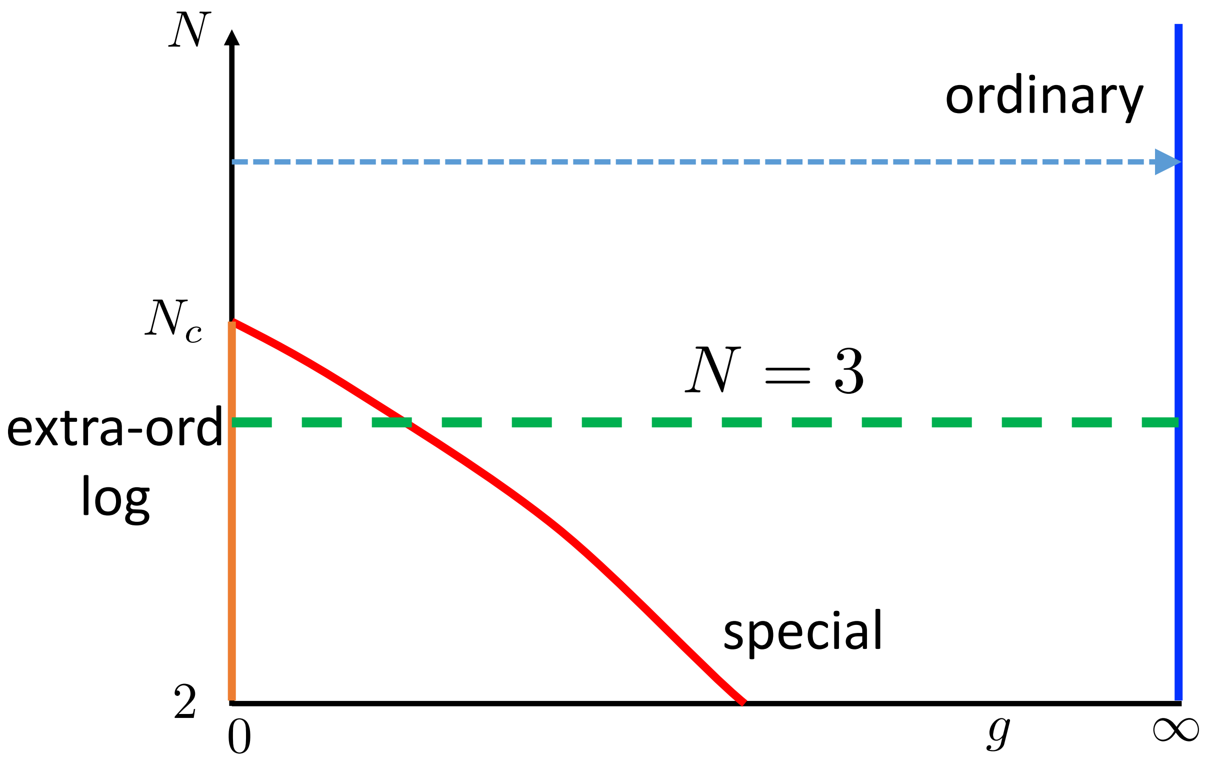
L. Weber and S. Wessel, 2019

Quantum models



- Controlled by special transition? $(\Delta_{\vec{n}})_{spec} \approx 0.26$

$$\nu_{spec}^{-1} \approx 0.36$$



C. Ding, L. Zhang and W. Guo, 2018
C.-M. Jian, Y. Xu, X.-C. Wu, C. Xu, 2020

S = ½ vs S = 1?

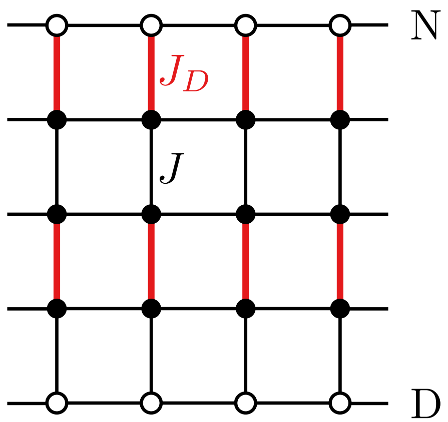
$$S = \frac{1}{2g}\int d^2\mathrm{x}\ (\partial_\mu \vec{n})^2 + \frac{i\theta}{4\pi}\int d^2\mathrm{x}\,\vec{n}\cdot(\partial_x \vec{n}\times\partial_\tau \vec{n}), \qquad \theta = 2\pi S$$

Haldane, 1983

$$S_{skyrm} = \frac{4\pi|m|}{g}$$

- Supression: $e^{-\frac{4\pi}{g_*}} \approx e^{-\frac{2}{\Delta_n}} \approx e^{-8}$

VBS



$$\vec{S}_i \cdot \vec{S}_{i+1} \sim (-1)^i V(x)$$

$$S = \frac{1}{2g} \int d^2\mathrm{x}\ (\partial_\mu \vec{n})^2 + \frac{i\theta}{4\pi} \int d^2\mathrm{x}\, \vec{n} \cdot (\partial_x \vec{n} \times \partial_\tau \vec{n})$$

$$V(x) \sim \vec{n} \cdot (\partial_x \vec{n} \times \partial_\tau \vec{n})$$

$$\Delta_V \approx 2$$

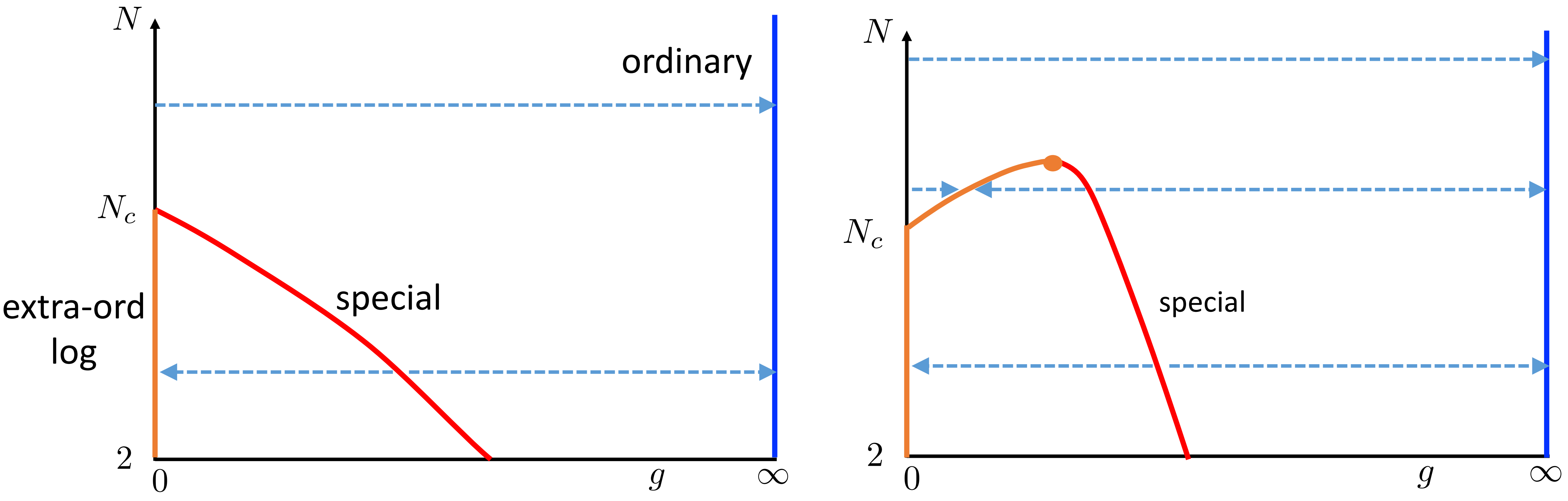
- Numerics:

S = 1/2: $\Delta_V < 2 \quad (\Delta_V \approx 1.4)$

S = 1: $\Delta_V \approx 2$

L. Weber and S. Wessel (2020)

Conclusion



- Monte-Carlo (classical/quantum)
- Conformal bootstrap
- Special transition with N = 2

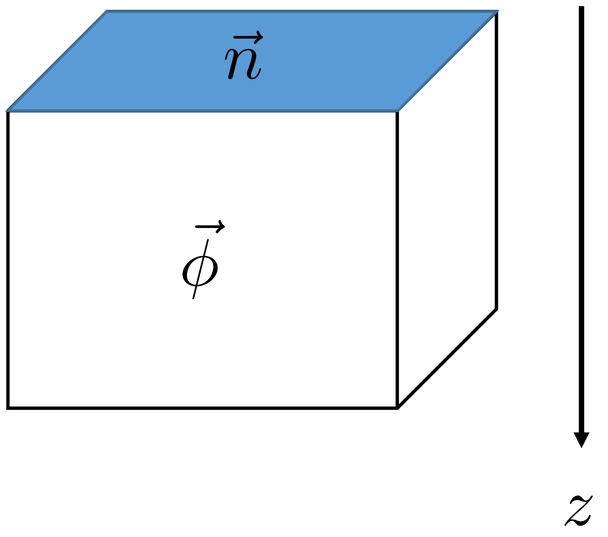
RG details

$$S_n = \frac{1}{2g}\int d^2\mathrm{x}\ (\partial_\mu \vec{n})^2$$

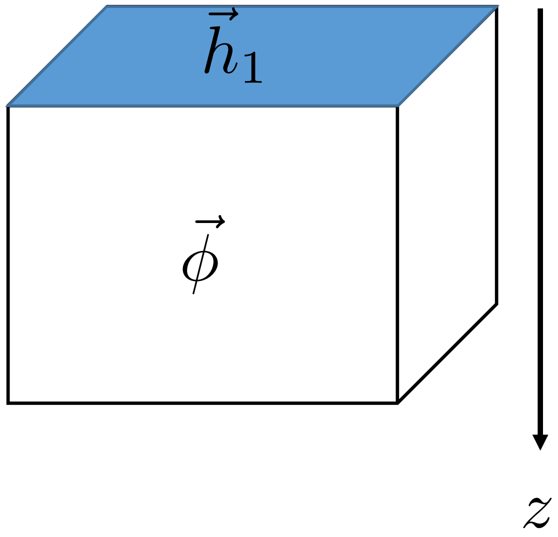
$$S = S_{ord}[\vec{\phi}] + S_n - \tilde{s}\int d^2\mathrm{x}\ \vec{n}\cdot\vec{\phi}(\vec{\mathrm{x}}, z=0)$$

$$\vec{n} = (\vec{\pi}, \sqrt{1-\vec{\pi}^2})$$

- $g = 0\,,\quad \vec{n} = (\vec{0}, 1)$ - flows to normal universality class



Normal universality class



$$\phi_N(\vec{\mathrm{x}}, z) \sim \frac{A_\sigma}{z^{\Delta_\phi}} + \mu_\sigma z^{3-\Delta_\phi} \hat{\sigma}(\vec{\mathrm{x}}) + \ldots, \quad z \to 0$$

$$\phi_i(\vec{\mathrm{x}}, z) \sim \mu_\phi z^{2-\Delta_\phi} \hat{\phi}_i(\vec{\mathrm{x}}) + \ldots, \quad z \to 0, \qquad i = 1 \ldots N-1$$

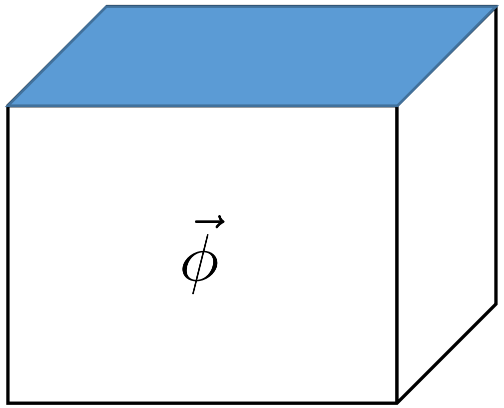
$$\langle O^a(x) O^b(y) \rangle = \frac{\delta^{ab}}{|x-y|^{2\Delta_O}}, \qquad \langle \hat{O}^a(\mathrm{x}) \hat{O}^b(\mathrm{y}) \rangle = \frac{\delta^{ab}}{|\mathrm{x}-\mathrm{y}|^{2\Delta_{\hat{O}}}},$$

$$\Delta_{\hat{\phi}_i} = 2 \qquad \Delta_{\hat{\sigma}} = 3$$

Bray and Moore, 1977;
Burkhardt and Cardy, 1987.

From ordinary to normal

$$\vec{n} = (\vec{\pi}, \sqrt{1 - \vec{\pi}^2})$$



$$S = S_{ord}[\vec{\phi}] + S_n - \tilde{s} \int d^2\mathrm{x}\ \vec{n} \cdot \vec{\phi}(\vec{\mathrm{x}}, z = 0)$$

$$\to S_{norm}[\vec{\phi}] + S_n - s \int d^2\mathrm{x}\ \pi_i \hat{\phi}_i$$

$$\phi_i(\vec{\mathrm{x}}, z) \sim \mu_\phi z^{2-\Delta_\phi} \hat{\phi}_i(\vec{\mathrm{x}}) + \dots, \quad z \to 0, \qquad i = 1 \dots N-1$$

(normal)

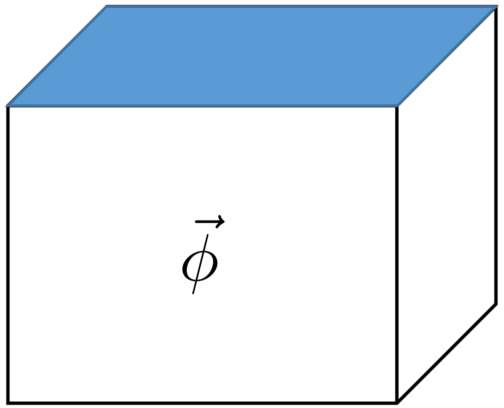
$$\phi_N(\vec{\mathrm{x}}, z) \sim \frac{A_\sigma}{z^{\Delta_\phi}} + \mu_\sigma z^{3-\Delta_\phi} \hat{\sigma}(\vec{\mathrm{x}}) + \dots, \quad z \to 0$$

$$\boxed{s = \frac{1}{\pi} \frac{A_\sigma}{\mu_\phi}}$$

$\delta L \sim \hat{\sigma}\sqrt{1 - \vec{\pi}^2}$ - irrelevant, $\Delta_{\hat{\sigma}} = 3$

Restoring O(N) symmetry

$$\vec{n} = (\vec{\pi}, \sqrt{1-\vec{\pi}^2})$$



$$S = S_{norm}[\vec{\phi}] + S_n - s \int d^2\mathrm{x}\ \pi_i \hat{\phi}_i$$

$$\vec{\pi} = 0, \qquad \langle \phi_N(z) \rangle = \frac{A_\sigma}{z^{\Delta_\phi}}, \quad \langle \phi_i \rangle = 0$$

$$\vec{\pi} \neq 0, \qquad \langle \phi_i(z) \rangle \approx \frac{A_\sigma}{z^{\Delta_\phi}}\, \pi_i$$

$$\langle \phi_i(z) \rangle \approx s\, \pi_j \int d^2\mathrm{x}\ \langle \phi_i(z) \hat{\phi}_j(\mathrm{x}) \rangle_{norm}$$

$$\langle \phi_i(\mathrm{x}, z) \hat{\phi}_j(\mathrm{x}') \rangle_{norm} = \mu_\phi \delta_{ij} \frac{z^{2-\Delta_\phi}}{(|\mathrm{x}-\mathrm{x}'|^2 + z^2)^2}$$

$$\boxed{s = \frac{1}{\pi} \frac{A_\sigma}{\mu_\phi}}$$

Amplitudes

$$A_\sigma^2 \approx N + 0.678 + O(N^{-1})$$

$$\mu_\phi^2 \approx 2\left(1 + \frac{0.678}{N}\right) + O(N^{-2})$$

$$s^2 = \frac{N}{2\pi^2} + O\left(N^{-1}\right)$$

MM + Ohno, Okabe, 1983

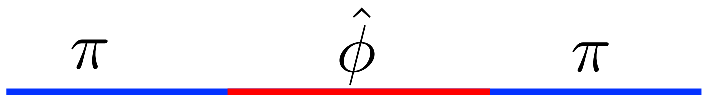
- $4 - \epsilon$: P. Dey, T. Hansen, M. Shpot, 2020.

RG

$$S = S_{norm}[\vec{\phi}] + S_n - s\int d^2\mathrm{x}\ \pi_i \hat{\phi}_i$$

$$s = \frac{1}{\pi}\frac{A_\sigma}{\mu_\phi}$$

$$S_n = \frac{1}{2g}\int d^2x \left((\partial_\mu \vec{\pi})^2 + \frac{1}{1-\vec{\pi}^2}(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 \right)$$



$$\langle \hat{\phi}_i(\mathrm{x})\hat{\phi}_j(\mathrm{x}')\rangle = \frac{\delta_{ij}}{|\mathrm{x}-\mathrm{x}'|^4}$$

$$\frac{dg}{d\ell} = -\alpha g^2$$

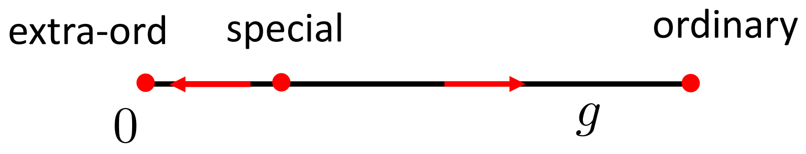
$$\alpha = \frac{\pi s^2}{2} - \frac{N-2}{2\pi}$$

$$\eta_n = \frac{N-1}{2\pi} g$$

Check: d = 3 + ϵ

$$\frac{dg}{d\ell} = -\epsilon g - \alpha g^2 \qquad \eta_n = \frac{N-1}{2\pi} g \qquad \alpha = \frac{\pi s^2}{2} - \frac{N-2}{2\pi} \qquad s = \frac{1}{\pi}\frac{A_\sigma}{\mu_\phi}$$

$$N > N_c, \quad \alpha < 0$$

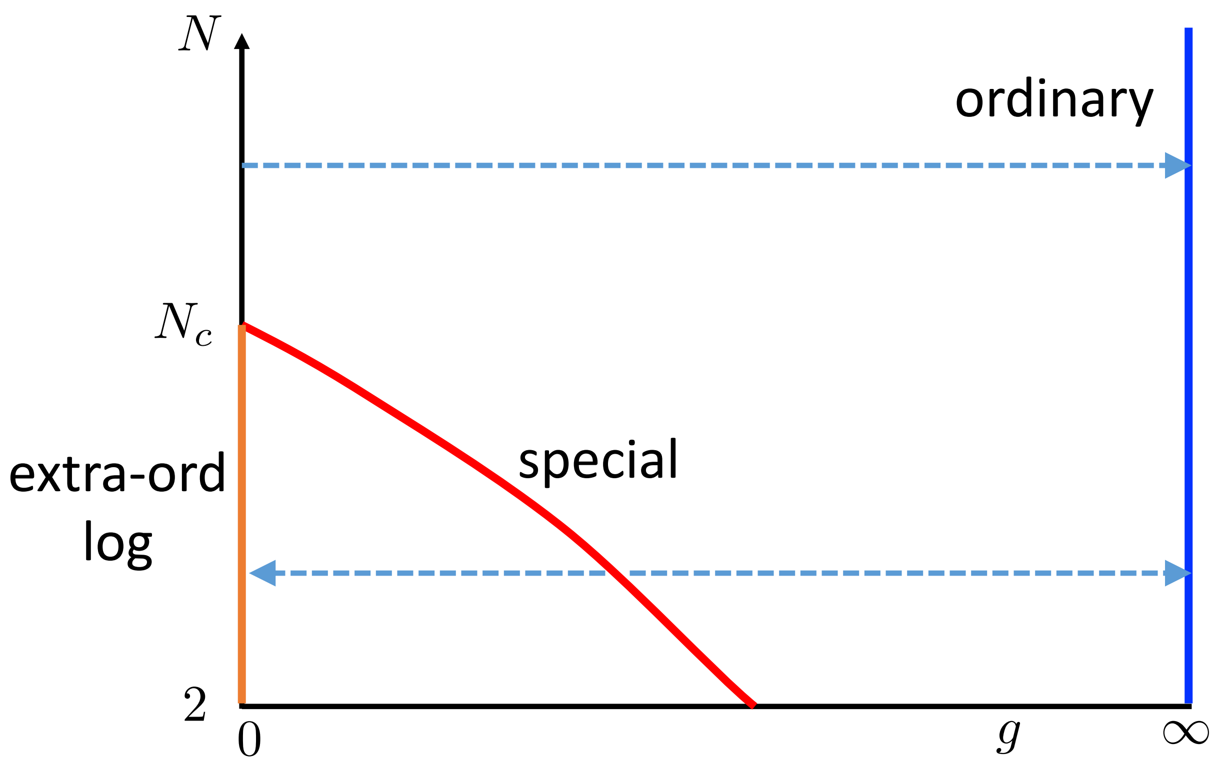


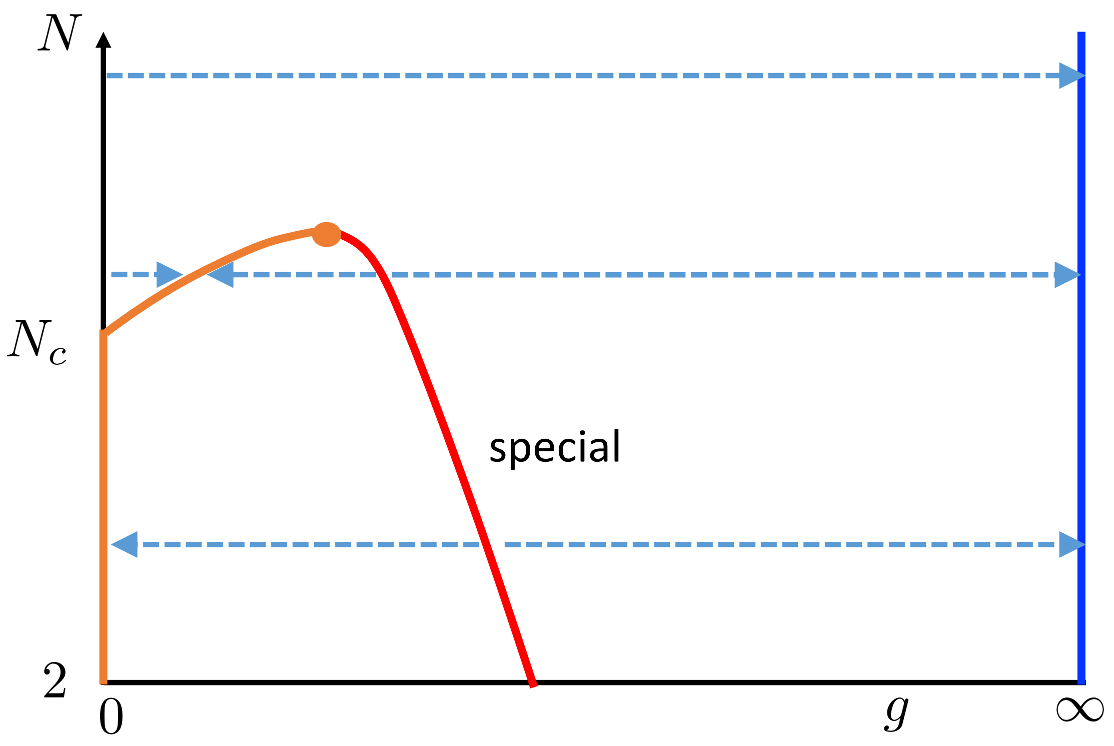
$$g_*^{spec} \approx \frac{\epsilon}{|\alpha|}, \qquad \Delta_n^{spec} = \frac{\eta_n(g_*)}{2}$$

$$N \to \infty: \qquad \alpha \to -\frac{N-4}{4\pi} \qquad \Delta_n^{spec} = \epsilon\left(1 + \frac{3}{N} + O(N^{-2})\right) + O(\epsilon^2)$$

Agrees with Ohno, Okabe 1983

Conclusion





- Monte-Carlo (classical/quantum)
- Bootstrap $s = \dfrac{1}{\pi}\dfrac{A_\sigma}{\mu_\phi}$
- Special transition with N = 2

Thank you!